NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE

(Accredited by NAAC, Approved by AICTE New Delhi, Affiliated to APJKTU)

Pampady, Thiruvilwamala(PO), Thrissur(DT), Kerala 680 588

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

COURSE MATERIALS

MAT 101 LINEAR ALGEBRA AND CALCULUS

VISION OF THE INSTITUTION

To mould our youngsters into Millennium Leaders not only in Technological and Scientific Fields but also to nurture and strengthen the innate goodness and human nature in them, to equip them to face the future challenges in technological break troughs and information explosions and deliver the bounties of frontier knowledge for the benefit of humankind in general and the down-trodden and underprivileged in particular as envisaged by our great Prime Minister Pandit Jawaharlal Nehru

MISSION OF THE INSTITUTION

To build a strong Centre of Excellence in Learning and Research in Engineering and Frontier Technology, to facilitate students to learn and imbibe discipline, culture and spirituality, besides encouraging them to assimilate the latest technological knowhow and to render a helping hand to the under privileged, thereby acquiring happiness and imparting the same to others without any reservation whatsoever and to facilitate the College to emerge into a magnificent and mighty launching pad to turn out technological giants, dedicated research scientists and intellectual leaders of the society who could prepare the country for a quantum jump in all fields of Science and Technology

ABOUT DEPARTMENT

- Established in: 2002
- Course offered: B.Tech COMPUTER SCIENCE AND ENGINEERING

: M.TECH COMPUTER SCIENCE AND ENGINEERING

:M.TECH CYBER SECURITY

- Approved by AICTE New Delhi and Accredited by NAAC
- Affiliated to the University of . A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Producing Highly Competent, Innovative and Ethical Computer Science and Engineering Professionals to facilitate continuous technological advancement

DEPARTMENT MISSION

M1: To Impart Quality Education by creative Teaching Learning Process

M2: To Promote cutting-edge Research and Development Process to solve real world problems with emerging technologies.

M3: To Inculcate Entrepreneurship Skills among Students

M4: To cultivate Moral and Ethical Values in their Profession

PROGRAMME EDUCATIONAL OBJECTIVES

PEO1: Graduates will be able to Work and Contribute in the domains of Computer Science and Engineering through lifelong learning.

PEO2: Graduates will be able to Analyse, design and development of novel Software Packages, Web Services, System Tools and Components as per needs and specifications.

PEO3: Graduates will be able to demonstrate their ability to adapt to a rapidly changing environment by learning and applying new technologies.

PEO4: Graduates will be able to adopt ethical attitudes, exhibit effective communication skills, Teamwork and leadership qualities.

C101.1	Solve the convergent test in mathematical series
C101.2	Acquire the basic knowledge about three dimensional spaces and integral calculus of functions of more than one variables
C101.3	Understand about partial derivatives and its applications
C101.4	Solve problems in calculus of vector valued functions
C101.5	Apply multiple integrals to find area and volume
C101.6	Evaluate surface and volume integrals

PROGRAM OUTCOMES (PO'S)

After the successful completion of the Couse, B.Tech. Computer Science and Engineering, **Graduates can able to**

PO1: Engineering Knowledge: Apply the knowledge of Mathematics, Science, to solve complex engineering problems related to Design, Development, Testing and Maintenance of Software and System Tools

PO2: Problem Analysis: Identify, Analyse and Formulate complex problems to achieve significant conclusions by applying Mathematics, Natural Sciences and Computer Science and Engineering Principles and Technologies.

PO3: Design/Development of solutions: Design and construct software system, programme, component or process to meet the desired needs within the realistic constraints.

PO4: Conduct investigations of complex problems: Use research based knowledge and research methods to perform Literature Survey, design experiments for complex problems in designing, developing and maintaining computing systems, collect data from experimental outcome, analyse and interpret the interesting patterns and to provide effective conclusions.

PO5: Modern tool usage: Create, select and apply appropriate state-of-the-art Tools and Techniques in designing, developing, testing and validating Computing Systems, Tools and Components.

PO6: The engineer and society: Assess the societal, health, security, legal and cultural issues that might arise during Professional Practice in Computer Science and Engineering.

PO7: Environment and sustainability: Demonstrate the knowledge of sustainable development of Software, Components, Tools, Computing Systems and Solutions with an understanding of the impact of these engineering solutions on society and environment.

PO8: Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice of Computer Science and Engineering.

PO9: Individual and Team Work: Function effectively as an individual, and as a member or leader in multi-disciplinary teams, and strive to achieve common goals.

PO10:Communication: Communicate effectively with engineering community and society and be able to comprehend and write effective reports and documents, make effective presentations and give and receive clear instructions.

PO11:Project Management and Finance: Apply knowledge of the Engineering and Management principles to one's own work, as a member and leader in a team, to manage projects in Multidisciplinary Teams.

PO12:Life-long learning: Recognize the need for lifelong learning to cope up with the rapidly emerging Cutting Edge Technologies in Computer Science and Engineering and its allied Engineering application domains.

CO'S	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C101.1	3	**3**	**3**	**3**	-	-	-	-	-	-	-	**1**
C101.2	3	**3**	**3**	**3**	-	-	-	-	-	-	-	**1**
C101.3	3	**3**	**3**	**3**	-	-	-	-	-	-	-	**1**
C101.4	3	**3**	**3**	**3**	-	-	-	-	-	-	-	**1**
C101.5	3	**3**	**3**	**3**	-	-	-	-	-	-	-	**1**
C101.6	3	**3**	**3**	**3**	-	-	-	-	-	-	-	**1**

HIGH	**3**
MODERATE	**2**
LOW	**1**
NIL	**-**

PROGRAM SPECIFIC OUTCOMES (PSO'S)

1). **PSO1: Analysis Skills:** Ability to Formulate and Simulate Innovative Ideas to provide software solutions for Real-time Problems.

2). **PSO2: Design Skills :** Ability to Analyse and design various methodologies for facilitating development of high quality System Software Tools and Efficient Web Design Models with a focus on performance optimization.

3). **PSO3: Product Development :** Ability to Apply Knowledge for developing Codes and integrating hardware/software products in the domains of Big Data Analytics, Web Applications and Mobile Apps

CO'S	PSO1	PSO2	PSO3
C101.1			2
C101.2		3	
C101.3	3	2	
C101.4	2	2	
C101.5	2	2	
C101.6	2	2	
C101	2.25	2.2	2

SYLLABUS

COURSE NO.	COURSE NAME	CREDITS	YEAR OF INTRODUCTION
MA 101	CALCULUS	4	2016

Course Objectives

In this course the students are introduced to some basic tools in Mathematics which are useful in modelling and analysing physical phenomena involving continuous changes of variables or parameters. The differential and integral calculus of functions of one or more variables and of vector functions taught in this course have applications across all branches of engineering. This course will also provide basic training in plotting and visualising graphs of functions and intuitively understanding their properties using appropriate software packages.

Syllabus

Single Variable Calculus and Infinite series, Functions of more than one variable, Partial derivatives and its applications, Calculus of vector valued functions, Multiple Integrals.

Expected outcome

At the end of the course the student will be able to (i) check convergence of infinite series (ii) find maxima and minima of functions two variables (iii) find area and volume using multiple integrals (iv) apply calculus of vector valued functions in physical applications and (v) visualize graphs and surfaces using software or otherwise.

Text Books

(1)Anton, Bivens, Davis: Calculus, John Wiley and Sons, 10th ed

(2)Thomas Jr., G. B., Weir, M. D. and Hass, J. R., Thomas' Calculus, Pearson

References:

1. Sengar and Singh, Advanced Calculus, Cengage Learning, Ist Edition
2. Erwin Kreyszig, Advanced Engineering Mathematics, Wiley India edition, 10th ed.
3. B. S. Grewal, Higher Engineering Mathematics, Khanna Publishers, New Delhi.
4. N. P. Bali, Manish Goyal, Engineering Mathematics, Lakshmy Publications
5. D. W. Jordan, P Smith. Mathematical Techniques, Oxford University Press, 4th

Edition.

6. A C Srivastava, P K Srivasthava, Engineering Mathematics Vol 1. PHI Learning Private Limited, New Delhi.

	COURSE NO: MA101	L-T-P:3-1-0	
	COURSE NAME: CALCULUS	CREDITS:4	
MODULE	CONTENT	HRS	END SEM. MARK %
I	Single Variable Calculus and Infinite series (Book I –sec 9.3,9.5,9.6,9.8) Basic ideas of infinite series and convergence - .Geometric series- Harmonic series-Convergence tests-comparison, ratio, root tests (without proof). Alternating series- Leibnitz Test-Absolute convergence, Maclaurins series-Taylor series - radius of convergence. (For practice and submission as assignment only: Sketching, plotting and interpretation of hyperbolic functions using suitable software. Demonstration of convergence of series bysoftware packages)	9	15%
II	Partial derivatives and its applications(Book I –sec. 13.3 to 13.5 and 13.8) Partial derivatives–Partial derivatives of functions of more than two variables - higher order partial derivatives - differentiability, differentials and local linearity -	5	15%
	The chain rule – Maxima and Minima of functions of two variables - extreme value theorem (without proof)-relative extrema .	4	

FIRST INTERNAL EXAM			
III	Calculus of vector valued functions(Book I-12.1,12.2,12.4&12.6,13.6 &13.7) Introduction to vector valued functions-parametric curves in 3-space Limits and continuity – derivatives - tangent lines – derivative of dot and cross product-definite integrals of vector valued functions- unit tangent-normal- velocity-acceleration and speed–Normal and tangential components of acceleration. Directional derivatives and gradients-tangent planes and normal vectors (For practice and submission as assignment only: Graphing parametric curves and surfaces using software packages)	 3 3 3	15%
IV	Multiple integrals (Book I-sec. 14.1, 14.2, 14.3, 14.5) Double integrals- Evaluation of double integrals – Double integrals in non-rectangular coordinates- reversing the order of integration- Area calculated as a double integral- Triple integrals(Cartesian co ordinates only)- volume calculated as a triple integral- (applications of results only)	 4 2 2 2	15%
SECOND INTERNAL EXAM			
	Topics in vector calculus (Book I-15.1, 15.2, 15.3) Vector and scalar fields- Gradient fields –	 2	

V	conservative fields and potential functions –	2	20%
	divergence and curl - the operator - the Laplacian ^{2} ,	2	
	Line integrals - work as a line integral-	2	
	independence of path-conservative vector field –	2	
	(For practice and submission as assignment only: graphical representation of vector fields using software packages)		
VI	Topics in vector calculus (continued)		20%
	(Book I sec., 15.4, 15.5, 15.7, 15.8)		
	Green's Theorem (without proof- only for simply connected region in plane),	2	
	surface integrals –	2	
	Divergence Theorem (without proof for evaluating surface integrals) ,	3	
	Stokes' Theorem (without proof for evaluating line integrals)	3	
	(All the above theorems are to be taught in regions in the rectangular co ordinate system only)		
END SEMESTER EXAM			

Open source software packages such as gnuplot, maxima, scilab ,geogebra or R may be used as appropriate for practice and assignment problems.

TUTORIALS: Tutorials can be ideally conducted by dividing each class in to three groups. Prepare necessary materials from each module that are to be taught using computer. Use it uniformly to every class.

QUESTION BANK

MODULE I

1. Show that the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ is convergent
2. Test the convergence of $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \cdots \ldots \ldots$
3. Check whether the series $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ converges or not
4. Determine whether the series $\sum_{k=1}^{\infty} (\frac{3}{4})^{k+2}$ converges and if so find its sum
5. Test the nature of the series $\sum_{k=1}^{\infty} \frac{4k^3-6k+5}{8k^7+k-8}$
6. Check whether the series $\sum_{n=1}^{\infty} \frac{1}{5n-1}$ converges or not
7. Check whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ converges or not
8. Test the convergence of $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$
9. Examine the convergence of the series $\sum_{k=1}^{\infty} \frac{k^k}{k!}$
10. Find the radius of convergence and the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$
11. Use ratio test for absolute convergence to find whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, 2^k}{k!}$ Converges
12. Check whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, k^k}{k!}$ Absolutely convergent or not
13. Find the radius of convergence and the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x-5)^k}{5^k}$
14. Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} n!\, x^n$
15. Check whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n\,(2n-1)!}{3^n}$ Absolutely convergent or not
16. Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{2n+3}$
17. Show that the series $\sum_{k=1}^{\infty} (\frac{1}{2})^k$ converges and $\sum_{k=1}^{\infty} (-1)^k$ diverges
18. Find the Taylor series of $\frac{1}{x}$ about $x = 1$
19. Find the Maclaurin's series for $\frac{1}{1-x}$
20. Find Maclaurin series for the function xe^x
21. Find the Taylor series expansion of $\log \cos x$ about the point $x = \frac{\pi}{3}$
22. Determine the Taylor series expansion of $f(x) = \sin x \ at\ x = \frac{\pi}{4}$
23. Find the Maclaurin series for $\cos x$ and also find $\cos 1$, calculate the absolute the error.
24. Determine whether the series $\sum_{k=1}^{\infty} \frac{5}{4^k}$ converges. If so find sum
25. Determine whether the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\ \ k+1}{k(k+4)}$ is absolutely convergent

26. Find the Taylor series of $\frac{1}{x+2}$ about $x = 1$
27. Find the interval of convergence and the radius of convergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+1)^k}{k}$

MODULE II

1. Let $w = 4x^2 + 4y^2 + z^2$ $where\ x = \rho sin\emptyset cos\theta$ $, y = \rho sin\emptyset sin\theta$ $z = \rho cos\emptyset$ find $\frac{\partial w}{\partial \rho}$ using chain rule
2. If $f(x, y) = x^2y^3 + x^4y$ find f_{xy}
3. If x^y $then\ find$ $\frac{\partial^2 z}{\partial x \partial y}$
4. Compute the differential $dz\ of\ the\ function\ z = tan^{-1}(xy)$
5. Find the slope of the surface $z = \sqrt{3x + 2y}$ $in\ the\ y - direction\ at\ the\ point\ (4,2)$
6. Find the derivative of $w = x^2 + y^2$ with respect to 't' along the path $x = at^2, y = 2at$
7. Given $z = e^{xy}$ $x = 2u + v, y = \frac{v}{u}$ $find$ $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$
8. Let $f\ be\ a\ differentiable\ function\ of\ three\ variable\ and\ suppose\ that\ w = f(x - y, y - z, z - x)\ Prove\ that$ $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$
9. Use chain rule find $\frac{dw}{ds}$ $at\ s = \frac{1}{4}$ $if\ w = r^2$-r tanθ , $r = \sqrt{s}$, $\theta = \pi s$
10. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ $show\ that$ $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2$ $u = \frac{-9}{(x+y+z)^2}$
11. If $u = \frac{x^3+y^3}{x-y}$ Find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$
12. $If\ w = 3xy^2z^3$, $y = 3x^2 + 2$, $z = \sqrt{x - 1}$ $find$ $\frac{dw}{dx}$ and $\frac{dw}{dy}$
13. Locate all relative maxima, relative minima and saddle point if any of $f(x, y) = y^2 + xy + 4y + 2x + 3$
14. Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ $at\ the\ point\ (3,4)$. Compare the error in approximating $f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}$ $by\ L(3.04,3.98)$ with the distance between the points (3,4) and (3.02,3.98)
15. A function $f(x, y) = x^2 + y^2$ is given with a local linear approximation $L(x, y)\ 2x + 4y - 5$ to f(x,y) at a point P. Determine the point P
16. Find the absolute extrema of the function $f(x, y) = xy - 4x\ of\ R$ where R is the triangular region with the vertices (0,0), (0,4) and (4,0)
17. Locate all relative extrema and saddle points of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$
18. Find the maximum and minimum values of $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
19. Let $L(x, y)$ denote the local linear approximation to $f(x, y) = \frac{x+y}{y+z}$ $at\ the\ point\ P(1,1,1)$. Compare the error in approximating $Q(-0.99,0.99,0.01)$ with the distance PQ
20. Find the slope of the surface $z = xe^{-y} + 5y$ $in\ the\ y - direction\ at\ the\ point\ (4,0)$
21. Show that the function $f(x, y) = e^x \sin y + e^y\ cosx$ satisfies the Laplace's equation $f_{xx} + f_{yy} = 0$

22. Let $L(x, y)$ denote the local linear approximation to $f(x, y) = xyz$ at the point $P(1,2,3)$. Also compare the error in approximating $Q(1.001,2.002,3.003)$ with the distance PQ.
23. Locate all relative extrema and saddle points of $f(x, y) = 2xy - x^3 - y^2$
24. If u=f($\frac{x}{y}$,$\frac{y}{z}$,$\frac{z}{x}$) prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

MODULE III

1. Find velocity , acceleration and speed of a particle moving along the curve $x = 1 + 3t, y = 3 - 4t, z = 1 + 3t \ at \ t = 2$
2. A particle moves along a circular helix in 3-space so that its position vector at time t is $r(t) = 4cos\pi ti + 4sin\pi tj + tk$. Find the displacement of the particle during the interval $1 \leq t \leq 5$
3. Find $y(t)$ $where\ y''(t) = 12t^2i - 2tj$, $y(0) = 2i - 4j$, $y'(0)$=0
4. Find the directional derivative of $f(x, y) = e^x secy\ at\ P(0,\frac{\pi}{4})$ in the direction of PQ where Q is the origin
5. Evaluate $\int_1^9 \frac{t}{2}i + \left(t - \frac{1}{2}\right)j\ dt$
6. Find $\frac{du}{dt}$ $if\ U = (3ti + 5t^2j + 6k).(t^2i + 2tj + tk)$
7. The temperature in degree Celsius at a point in the (x,y) plane is T(x,y)=$\frac{xy}{1+x^2+y^2}$. Find the rate of change of temperature at (1,1) in the direction $2i - j$
8. Let $f(x, y) = x^2e^y$. Find the maximum value of a directional derivative at (-2,0) and find the unit vector in the direction in which the maximum value occur
9. Find the domain of $r(t) =< \sqrt{5t + 1}, t^2 >, t_0 = 1\ and find\ r(t_0)$
10. $If\ F(t) has\ a\ constant\ direction, prove\ that\ FX\frac{df}{dt} = 0$
11. $Let\ r = xi + yj + zk\ and\ \boldsymbol{r} = |r| then\ prove\ that\ \nabla f(r) = \frac{f'(r)}{\boldsymbol{r}}r$

12 Find the directional derivative of $f = x^2y - yz^3 + z$ at (1,-2,0) in the direction of $a = 2i + j + 2k$

13. Find the directional derivative of $f(x, y, z) = x^3z - yx^2 + z^2\ at\ P(2, -1,1)$ in the direction of $3i - j + 2k$

MODULE IV

1. Evaluate $\int_1^a \int_1^b \frac{dydx}{xy}$
2. The line $y = 2 - x$ and the parabola $y = x^2$ intersects at the points (-2,4) and (1,1). If R is the region enclosed by $y = 2 - x$ and $y = x^2$ then find $\iint_R y\ dA$
3. Find the area bounded by the $x - axis$,$y = 2x\ and\ x + y = 1$ using double integration
4. Sketch the region of integration and evaluate the integral $\int_1^2 \int_y^{y^2} dxdy$ by changing the order of integration.
5. Sketch the region of integration and evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)dydx$

6. By changing the order of integration evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\ dydx$
7. Evaluate $\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$
8. Evaluate $\iint_R \frac{\sin x}{x}\ dA$ where R is the triangular region bounded by $x - axis\ , y = x, and\ x = 1$
9. Find the area of the region R enclosed between the parabola $y = \frac{x^2}{2}\ and\ the\ line\ y = 2x$
10. Evaluate $\iint_R\ y\, dA$ where R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$
11. Change the order of integration and evaluate $\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\ dxdy$
12. Find the area bounded by the parabolas $y^2 = 4x\ and\ x^2\ = -\frac{y}{2}$
13. Evaluate $\iint_{\mathbf{R}}\ \mathbf{x^2\ dA}$ over the region **R** enclosed between $\mathbf{y} = \frac{\mathbf{16}}{\mathbf{x}}$, $\mathbf{y} = \mathbf{x}\, , \mathbf{and\ \ x = 8}$
14. Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes z=1 and z=5
15. Evaluate $\int_0^3 \int_0^2 \int_0^1 (xyz)dxdydz$
16. Evaluate $\int_0^3 \int_{y^2}^1 \int_0^{1-x} (x)dzdxdy$
17. find the volume bounded by the cylinder $x^2 + y^2 = 4$ the planes z=0 and y+z=3
18. Find the volume of the paraboloid of revolution $x^2 + y^2 = 4z\ cut\ off\ \ by\ the\ plane\ z = 4$
19. Using double integration, evaluate the area enclosed by the lines $x = 0,\ \ y = 0,\ \ \frac{x}{a}+\frac{y}{b} = 1$
20. Evaluate $\int_{-1}^2 \int_0^2 \int_0^1 (x^2 + y^2 + z^2)dxdydz$
21. If R is the region bounded by the parabolas $y = x^2\ \ and\ y^2 = x\ in\ the\ first\ quadrant,$ evaluate $\iint_R\ (x + y)dA$
22. Use a triple integral to find the volume of the solid within the cylinder $y = x^2$ and the planes $\qquad y + z = 4,\ \ \ z = 0$

MODULE V

1. Confirm that $\emptyset(x, y, z) = x^2 - 3y^2 + 4z^3$ is a potential function for $\boldsymbol{F}(x, y, z) = 2xi - 6yj + 12z^2k$
2. Find $div\ \boldsymbol{F}\ and\ curl\ \boldsymbol{F}$ of $\boldsymbol{F}(x, y, z) = x^2y\ i + 2y^3z\ j + 3zk$
3. Determine whether $\boldsymbol{F}(x, y) = 4yi + 4xj$ is a conservative vector field. If so find the potential function and the potential energy.
4. Show that $\boldsymbol{F}(x, y, z) = (2xy + z^3)i + x^2j + 3xz^2k$ is a conservative vector field. Also find its scalar potential.
5. Find the values of the constants $a, b, c\ so\ that\ \ \boldsymbol{F}(x, y, z) = (axy + bz^3)i + (3x^2 - cz)j + (3xz^2 - y)k$ may be irrotational. For these values of a, b, c find the scalar potential of $\boldsymbol{F}$

6. Find $curl\,\boldsymbol{F}$ at the point $(\mathbf{1},-\mathbf{1},\mathbf{1})$ $where\ \boldsymbol{F} = xz^3 i - 2x^2yzj + 2yz^4k$
7. The function $\emptyset(x,y,z) = xy + yz + xz$ is a potential for the vector field . Find the vector field $\boldsymbol{F}$
8. If $\boldsymbol{r} = xi + yj + zk$ then show that $\nabla^2(r^n) = n(n+1)r^{n-2}$, where r=|**r**|
9. Evaluate$\int_C \boldsymbol{F}.\boldsymbol{dr}$ $where\ \boldsymbol{F} = sinxi + cosxj$ where c is the curve $\boldsymbol{r(t)} = \pi i + tj \quad 0 \leq t \leq 2$
10. Find the work done by the force field $\boldsymbol{F}(x,y,z) = (x^2 + xy)i + (y - x^2y)j$ on the particle that moves along the curve $c{:}\, x = t, y = \frac{1}{t}$, $1 \leq t \leq 3$
11. Evaluate $\int \boldsymbol{F}.\boldsymbol{dr}$ $where$ $\boldsymbol{F} = yi - xj$ along the triangle joining $(0{,}0), (1{,}0) and\ (0{,}1)$
12. Show that $\int_A^B((2xy + z^3)i + x^2j + 3xz^2k)$ is independent of the path joining the points A and B
13. Evaluate the line integral $\int_C (xy + z^3)ds$ from $(1{,}0{,}0) to\ (-1{,}0,\pi)$ along the helix C that is represented by the parametric equations $x = cost,\ y = sint,\ z = t$
14. Evaluate the line integral $\int_C (y - x)dx + (x^2)\,dy$ along the curve $C:\ y^2 = x^3\ from\ (1,-1) to\ (1{,}1)$
15. Find the work done by the force field $\boldsymbol{F} = (\boldsymbol{x} + \boldsymbol{y})\boldsymbol{i} + \boldsymbol{xyj} - \boldsymbol{z^2k}$ along the line segment from $(0{,}0{,}0) to\ (1{,}3{,}1) and\ then\ to\ (2,-1{,}5)$
16. If $F(x,y,z) = x^2i - 3j + yz^2k \quad find\ div\ F$
17. Find the work done by the force field $F = xyi + yzj + zxk$ on a particle that moves along the curve $C{:}\ x = t,\ y = t^2, z = t^3 \quad 0 \leq t \leq 1$
18. If $\boldsymbol{r} = xi + yj + zk$ then show that $\nabla\ (\log r) = \frac{\boldsymbol{r}}{r^2}$ where r=|**r**|
19. Compute the line integral $\int_c (y^2dx - x^2dy)\ along\ the\ triangle$ whose vertices are (1,0), (0,1)and (-1,0)
20. $\int_c (ysinx\ dx - \cos x\ dy)$ is independent of the path and hence evaluate it from (0,1) and $(\pi,-1)$
21. $Examine\ whether\ F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ is a conservative field. If so, Find the potential function.
22. $Show\ that\ \nabla^2 f(r)$=2$\frac{f'(r)}{r} + f''(r)$, where $\boldsymbol{r} = \boldsymbol{xi} + \boldsymbol{yj} + \boldsymbol{zk}$,r = ||**r**||

MODULE VI

1. Using Green's theorem evaluate $\oint_C y\,dx + x\,dy\ , where\ C$ is the unit circle oriented counter clock wise.
2. Using Green's theorem evaluate $\oint_C x\,dy - ydx\ , where\ C$ is the unit circle $x^2 + y^2 = a^2$

3. If σ is any closed surface enclosing a volume V and F=$2x\,i + 2y\,j + 3z\,k$ using Divergence theorem show that $\iint_\sigma F.n\,ds = 7V$
4. If S is any closed surface enclosing a volume V and F=$x\,i + 2y\,j + 3z\,k$ using Divergence theorem show that $\iint_S F.n\,ds = 6V$

5. Using Green's theorem evaluate $\oint_C (e^x + y^2)dx + (e^y + x^2)dy$ *where* C is the boundary of the region between $y = x^2$ *and* $y = 2x$
6. Evaluate the surface $\iint_\sigma \frac{x^2+y^2}{y}$ ds over the surface σ represented by the vector valued function $r(u,v) = 2\cos v\,i + uj + 2\,sinvk \quad 1 \le u \le 3 \;, \quad 0 \le v \le \pi$
7. Using Divergence theorem evaluate $\iint_\sigma F.n\,ds$ where $F(x,y,z) = (x-z)i + (y-x)j + (2z-y)k$ σ is the surface of the cylindrical solid bounded by $x^2 + y^2 = a^2,\; z = 0, z = 1$
8. Determine whether the vector field $F(x,y,z) = 4(x^3 - x)i + 4(y^3 - y)j + 4(z^3 - z)k$ is free of sources and sinks. If it is not , locate them.
9. Verify Green's theorem for $\int_C (xy + y^2)dx + x^2dy$ $where\ C$ is bounded by $y = x\ and\ y = x^2$
10. Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ $where\ C$ is the boundary of the area of enclosed by the x-axis and the upper half of the circle $x^2 + y^2 = a^2$
11. Apply Stoke's theorem to evaluate $\int_C (x+y)dx + (2x-y)dy + (y+z)dz$ $where\ C$ is the boundary of the triangle with vertices (0,0,0), (2,0,0)and (0,3,0)
12. Use Divergence theorem to evaluate $\iint_\sigma F.n\,ds\ where\ F = xi + zj + yzk\ and\ S\ is\ the\ surface\ of\ the\ cube\ bounded\ by\ x = 0, x = 1,\ y = 0, y = 1, z = 0, z = 1$.Also verify the result by computing surface integral over S
13. State Divergence theorem . Also evaluate $\iint_\sigma F.n\,ds$. $Where\ F = axi + byj + czk\ and\ S\ is\ the\ surface\ of\ the\ sphere\ x^2 + y^2 + z^2 = 1$
14. Using line integral evaluate the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
15. Evaluate $\int_C e^x dx + 2y\,dy - dz\ \ where\ C\ is\ the\ curve\ x^2 + y^2 - 4z = 2$
16. Use Green's theorem to evaluate $\oint_C x\,cosy\,dx - y\,sinx\,dy \quad where\ C\ is\ the\ square\ with\ vertices\ (0,0), (\pi,0), (\pi,\pi) and\ (0,\pi)$

17. Use Stoke's theorem to evaluate the integral $\oint_C F.dr \quad where\ F = xy\ i + yz\,j + zx\ k \quad C\ is\ the\ triangle\ in\ the\ plane\ x + y + z = 1,\ with\ vertices\ (1,0,0),$ $(0,1,0)$ *and* $(0,0,1)$ with a counter clockwise orientation looking from the first octant towards the origin.
18. Use Gauss Divergence theorem to find the outward flux of vector field $F(x,y,z) = x^3i + y^3j + z^3k$ across the surface of the region enclosed by the circular cylinder $x^2 + y^2 = 9$ and the plane $Z = 0\ and\ Z = 2$
19. Find the workdone by the force field $F(x,y) = 4(e^x - y^3)i + 4(\cos y + x^3)j$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counter clockwise direction.
20. Evaluate the surface integral $\iint_S xz\,ds \qquad where\ S\ is\ the\ part\ of\ the\ plane\ \ x + y + z = 1\ that$ lies in the first octant.

21. Using Divergence theorem evaluate $\iint_{\sigma} F.n\,ds$ where $F(x,y,z) = (x^2 + \mathrm{y})\mathrm{i} + z^2\mathrm{j} + (e^y - \mathrm{z})\mathrm{k}$ and S is the surface of the rectangular solid bounded by the coordinate planes and the planes x=3 ,y=1, z=3
22. Apply Stoke's theorem to evaluate $\int_c F.dr$, $where$ $F = (x^2 - y^2)\mathrm{i} + 2\mathrm{xyj}$ and c is the rectangle in the xy plane bounded by the lines x=0 , y=0, x=a and y=b

(11)

Deepamol P.S.

I

Infinite Series (1)

(6) Definition :

An infinite series is an expression that can be written in the form $\sum_{K=1}^{\infty} u_K = u_1 + u_2 + u_3 + \cdots + u_K + \cdots$

The numbers $u_1, u_2, u_3 \cdots$ are called the terms of the series.

Eg:- consider the decimal $0.3333 \cdots$

This can be viewed as the infinite series

$0.3 + 0.03 + 0.003 + \cdots$ or $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

A sequence of partial sums of a series $\sum_{n=1}^{\infty} a_n$ is defined as the sequence $\{S_n\}$ where

$S_n = a_1 + a_2 + \cdots + a_n, \quad n = 1, 2, 3 \ldots$

eg:- consider the series $0.3 + 0.03 + 0.003 + \cdots$

then $S_1 = 0.3$

$S_2 = 0.3 + 0.03$

$S_3 = 0.3 + 0.03 + 0.003$

$\cdots$

$S_n = (0.3 + 0.03 + 0.003 + \cdots + 0.000 \cdots 03)$

$= \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$

(18)

Convergence of infinite Series

(2) Let $\{S_n\}$ be the sequence of partial sums of the series $u_1+u_2+u_3+\cdots+u_k+\cdots$ If the sequence $\{S_n\}$ converges to a limit S, then the series is said to converge to S, and S is called the sum of the series. It is denoted by $S=\sum_{k=1}^{\infty}u_k$

If the sequence of partial sums diverges, then the series is said to diverge.

A divergent series has no sum.

Eg: Determine whether the series $1-1+1-1+\cdots$, converges or diverges. If it converges, find the sum.

Here $S_1=1$

$S_2=1-1=0$

$S_3=1-1+1=1$

$S_4=1-1+1-1=0$

Thus the sequence of partial sum is $1,0,1,0,1,0\ldots$

This is a divergent sequence.

Hence the given series is also divergent and

Consequently has no sum

Geometric Series

Infinite series of the ~~sum~~ form $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^k + \dots$ $(a \neq 0)$ is called geometric series. The number 'r' is called the ratio for the series

Eg: * $1 + 2 + 4 + 8 + \dots + 2^k + \dots$ Here $a = 1$ & $r = 2$

* $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \dots + (-1)^{k+1} \frac{1}{2^k} + \dots$

Here $a = 1/2$ $r = -1/2$

Thereorem

A geometric series $\sum_{k=0}^{\infty} ar^k = a + ar + \dots + ar^k + \dots$ $(a \neq 0)$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

* Determine whether the series converges, and if so find its sum

(1) $\sum_{k=0}^{\infty} \frac{5}{4^k}$

$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \dots + \frac{5}{4^k} + \dots$ Geometric series

Here $a = 5$ & $r = \frac{1}{4}$

Since $|r| = \left|\frac{1}{4}\right| < 1$, the given G.S is conv

and Sum is $\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{20}{3}$

(2) Find the rational number represented by repeat decimal $0.784784784\ldots$

$\rightarrow$ $0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 + \ldots$

$= \frac{784}{10^3} + \frac{784}{10^6} + \frac{784}{10^9} + \ldots$

This is a geometric series with $a = \frac{784}{10^3}$, $r = \frac{1}{10}$

Here $|r|<1$ ∴ The G. Series Converges.

Sum $= \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{.784}{.999} = \frac{784}{999}$

* Find all values of x for which the series

$3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \ldots + \frac{3(-1)^k}{2^k} x^k \ldots$

converges and find the sum of the series for those values of x.

$\rightarrow$ This is a geometric series with $a = 3$, & $r = -$

converges if $\left|-\frac{x}{2}\right|<1$, or equivalently when $|x|<2$

when the series converges its sum is,

$$\sum_{K=0}^{\infty} 3\left(\frac{-x}{2}\right)^K = \frac{3}{1-\left(-\frac{x}{2}\right)} = \frac{6}{2+x}$$

Harmonic Series

An infinite series of the form $\sum_{K=1}^{\infty} \frac{1}{K}$ - $1+\frac{1}{2}+\frac{1}{3}+ \cdots$ is called Harmonic Series. This series is divergent

· Convergence Tests

I Comparison Test

Theorem :- Let $\sum_{K=1}^{\infty} a_K$ and $\sum_{K=1}^{\infty} b_K$ be series with nonnegative terms and suppose that

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3,\ \ldots,\ a_K \le b_K, \ldots$$

a) If $\sum b_K$ converges, then $\sum a_K$ also converges

b) If $\sum a_K$ diverges, then $\sum b_K$ also diverges

p-Series

An infinite Series $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ converges if $p > 1$ and diverges if $0 < p \le 1$

Problem

* Use the comparison test to determine whether the following Series converge or diverge

1) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$

→ consider the Series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ which is divergent (It is a p-series $p = 1/2 < 1$) Also $\frac{1}{\sqrt{k}-\frac{1}{2}} > \frac{1}{\sqrt{k}}$ for $k = 1, 2, \ldots \infty$

Hence by comparison test, the given Series is divergent.

2) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$

→ We have $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-Series (p ...

Also $\frac{1}{2K^2+K} < \frac{1}{2K^2}$ for $K = 1, 2 \ldots$

Hence by comparison test the given series is convergent. (7)

Limit comparison Test

Let Σa_K and Σb_K be series with positive terms and suppose that $\rho = \lim_{K\to\infty} \frac{a_K}{b_K}$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge

* use limit Comparison test determine whether the series is convergent or divergent

1) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}+1}$

Let $a_K = \frac{1}{\sqrt{K}+1}$ & $b_K = \frac{1}{\sqrt{K}}$ (Σb_K is a divergent series)

$$\rho = \lim_{K\to\infty} \frac{a_K}{b_K} = \lim_{K\to\infty} \frac{\sqrt{K}}{\sqrt{K}+1} = \lim_{K\to\infty} \frac{1}{1+\frac{1}{\sqrt{K}}}$$

$$= 1$$

ρ is finite and positive. Therefore by limit comparison test the given series diverges

(24)

2) $\sum_{K=1}^{\infty} \frac{1}{2K^2+K} = \sum_{K=1}^{\infty} \frac{1}{2K^2(1+\frac{1}{2K})}$

Let $\sum b_K = \sum \frac{1}{2K^2}$ which is convergent

$\rho = \lim_{K\to\infty} \frac{a_K}{b_K} = \lim_{K\to\infty} \frac{1}{1+1/2K} = 1$ finite & positive $\therefore$ by limit comparison test, the given series is convergent.

Limit comparison test

Let $\sum a_K$ and $\sum b_K$ be series with positive terms and suppose that $\rho = \lim_{K\to\infty} \frac{a_K}{b_K}$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge

* Use limit comparison test determine whether the series is convergent or divergent

1) $\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}+1}$

$\frac{1}{\sqrt{K}+1} = \frac{1}{\sqrt{K}(1+\frac{1}{\sqrt{K}})}$

$K^{1/2}$, $1/2 < 1$ PS

Let $a_K = \frac{1}{\sqrt{K}+1}$ & $b_K = \frac{1}{\sqrt{K}}$ ($\sum b_K$ is divergent)

$\rho = \lim_{K\to\infty} \frac{a_K}{b_K} = \lim_{K\to\infty} \frac{\sqrt{K}}{\sqrt{K}+1} = \lim_{K\to\infty} \frac{1}{1+}$

$= 1$

ρ is finite and +ve. Therefore by limit comp

Test the given series diverges.

$$2)\ \sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \sum_{k=1}^{\infty} \frac{1}{2k^2\left[1+\frac{1}{2k}\right]}$$

(9)

Let $\sum b_k = \sum \frac{1}{2k^2}$ which is convergent

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{1+1/2k} = 1 \text{, finite \& }$$

positive. $\therefore$ By limit comparison test, the given series is convergent.

$$(3)\ \sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$$

$$= \sum_{k=1}^{\infty} \frac{3k^3\left[1-\frac{2}{3k}+\frac{4}{3k^3}\right]}{k^7\left[1-\frac{1}{k^4}+\frac{2}{k^7}\right]}$$

Take $b_k = \frac{3k^3}{k^7} = \frac{3}{k^4}$

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{3}{k^4}$ converges (P series)

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1-2/3k+\frac{4}{3k^2}}{1-\frac{1}{k^4}+\frac{2}{k^7}}$$

$= 1$ finite & Non zero

$\therefore$ By limit comparison test, the given series is convergent.

Q. Test the convergence of the series $\sum_{k=1}^{\infty} \frac{1}{3^k+11}$

(10)

$$\frac{1}{3^k+11} < \frac{1}{3^k}$$

$\frac{1}{3^k}$ is a geometric series $a = \frac{1}{3}$ & $r = \frac{1}{3} < 1$

$\therefore \sum_{k=1}^{\infty} \frac{1}{3^k}$ is convergent.

Hence by comparison test $\sum_{k=1}^{\infty} \frac{1}{3^k+11}$ is also convergent.

* $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2-3k}}$.

$$a_k = \frac{1}{\sqrt[3]{8k^2-3k}} = \frac{1}{(8k^2-3k)^{1/3}}$$

$$= \frac{1}{(8k^2)^{1/3}\left[1-\frac{3k}{8k^2}\right]^{1/3}}$$

$$= \frac{1}{8^{1/3}k^{2/3}\left(1-\frac{3}{8k}\right)}$$

$$= \frac{1}{2k^{2/3}\left(1-\frac{3}{8k}\right)}.$$

Take $b_k = \frac{1}{2k^{2/3}}$ [$\sum b_k$ is a p series with $p<1$ Hence divergent

$$S = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{\left(1-\frac{3}{8k}\right)^{1/3}} = 1$$ finite &

Positive. Hence Σa_k is also divergent by limit comparison test.

(11)

* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$

$$a_k = \frac{1}{(2k+3)^{17}} = \frac{1}{k^{17}\left(2+\frac{3}{k}\right)^{17}}$$

Take $b_k = \frac{1}{k^{17}} \Rightarrow \Sigma b_k$ converges

$$S = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1}{(2+3/k)^{17}}$$

$$= \frac{1}{2^{17}}$$, finite and +ve

∴ By limit comparison test the given series Σa_k is also convergent.

L' Hospital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes the indeterminate forms $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$

and $f(x)$ & $g(x)$ have derivatives of all order then,

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided the limit exists.

Again if $\frac{f'(x)}{g'(x)}$ takes indeterminate forms, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$ provided the limit

exist finitely.

Eg:- $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

$$\Rightarrow \lim_{x \to 2} \frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x-2)} = \lim_{x \to 2} \frac{2x}{1} = \underline{\underline{4}}$$

Aliter

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = 2+2 = \underline{\underline{4}}$$

Note

* comparison test only applies to series with non-negative terms.

Ratio Test:-

Let $\sum u_k$ be a series with positive terms and suppose that, $\rho = \lim_{k \to \infty} \frac{u_{k+1}}{u_k}$.

[Try this test when u_k involves factorials or k^{th} power]

(i) if $\rho < 1$, the series converges

(ii) if $\rho > 1$ or $\rho = \infty$ the series diverges

(iii) if $\rho = 1$, the series may converge or diverge so that another test must be tried.

(1) Test whether the series converge or diverge

$$\sum_{K=1}^{\infty} \frac{1}{K!}$$

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$$\rho = \lim_{K\to\infty} \frac{U_{K+1}}{U_K} = \lim_{K\to\infty} \frac{\frac{1}{(K+1)!}}{1/K!} = \lim_{K\to\infty} \frac{K!}{(K+1)!}$$

$$= \lim_{K\to\infty} \frac{K!}{(K+1)K!}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} = 0 < 1$$

Hence the given series is convergent by Ratio Test.

(ii) $$\sum_{K=1}^{\infty} \frac{K}{2^K}$$

$$\rho = \lim_{K\to\infty} \frac{\frac{K+1}{2^{K+1}}}{K/2^K} = \lim_{K\to\infty} \frac{K+1}{K} \frac{2^K}{2^{K+1}} = \frac{1}{2} \lim_{K\to\infty} \frac{K+1}{K}$$

$$= \frac{1}{2} \lim_{K\to\infty} \frac{K(1+1/K)}{K}$$

$$= 1/2 < 1$$

Given series is convergent.

(iii) $$\sum_{K=1}^{\infty} \frac{K^K}{K!}$$

$$\rho = \lim_{K\to\infty} \frac{(K+1)^{K+1}}{(K+1)!} \cdot \frac{K!}{K^K}$$

$$= \lim_{K\to\infty} \frac{(K+1)^{K+1}}{K^K} \frac{K!}{(K+1)K!}$$

$$= \lim_{K \to \infty} \frac{(K+1)^K (K+1)}{K^K (K+1)}$$

$$= \lim_{K \to \infty} \left(\frac{K+1}{K} \right)^K = \lim_{K \to \infty} (1 + 1/K)^K$$

(iv)

$$= e > 1$$

$\therefore$ the Given Series is divergent.

(iii) $\sum_{K=3}^{\infty} \frac{(2K)!}{4^K}$

$$\lim_{K \to \infty} \frac{(2(K+1))!}{4^{K+1}} \cdot \frac{4^K}{(2K)!}$$

$$\lim_{K \to \infty} \frac{(2K+2)!}{(2K)!} \cdot \frac{1}{4}$$

$$= \lim_{K \to \infty} \frac{(2K+2)(2K+1)(2K)!}{(2K)!}$$

$= \underline{\underline{\infty}}$ $\therefore$ the Series diverges

(v) $\sum_{K=1}^{\infty} \frac{1}{2K-1}$

$$\rho = \lim_{K \to \infty} \frac{1}{2(K+1)-1} \cdot 2K-1$$

$$= \lim_{K \to \infty} \frac{1}{2K+1} \cdot 2K-1 = \lim_{K \to \infty} \frac{2K(1-\frac{1}{2K})}{2K(1+\frac{1}{2K})}$$

$= \underline{\underline{1}}$ Test fail

$*\ \sum_{k=1}^{\infty} \frac{1}{17k}$ $\quad$ $17k$

$$\rho = \lim_{k\to\infty} \frac{1}{17(k+1)}$$ (15)

$$= \lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{k}{k(1+1/k)} = 1$$

Test fail.

$*\ \sum_{k=1}^{\infty} \frac{1}{2k-1}$

$$\rho = \lim_{k\to\infty} \frac{1}{2(k+1)-1} \quad 2k-1$$

$$= \lim_{k\to\infty} \frac{2k-1}{2k+1} = \lim_{k\to\infty} \frac{2k(1-1/2k)}{2k\left(1+\frac{1}{2k}\right)} = \underline{1}$$

Test fail.

We have $\quad 2k-1 < 2k.$

$$\frac{1}{2k-1} > \frac{1}{2k}$$

Take $\sum b_k = \sum \frac{1}{2k} = \frac{1}{2}\sum \frac{1}{k}$ diverges

$\therefore$ By comparison test $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ also diverges

use the ratio test ~~the~~ to determine whether the series converges. If the test is inconclusive then say so

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* $\sum_{k=1}^{\infty} \frac{99^k}{k!}$

$$\rho = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(99)^{k+1}}{(k+1)!} \frac{k!}{(99)^k}$$

$$= \lim_{k\to\infty} \frac{99}{k+1} = 0 < 1$$

Hence the series is convergent by ratio test.

* $\sum_{k=1}^{\infty} \frac{4^k}{k^2}$

$$\rho = \lim_{k\to\infty} \frac{4^{k+1}}{(k+1)^2} \cdot \frac{k^2}{4^k}$$

$$= \lim_{k\to\infty} 4 \left(\frac{k^2}{k^2(1+\frac{1}{k})^2} \right)$$

$$= \lim_{k\to\infty} \frac{4}{(1+\frac{1}{k})^2} = 4 > 1$$

$\therefore$ Series diverges by ratio test.

* $\sum_{k=1}^{\infty} \frac{k!}{k^{99}}$

$$\rho = \lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{99}} \cdot \frac{k^{99}}{k!}$$

$= \infty$, Hence divergent

$$* \quad \sum_{k=1}^{\infty} \frac{k}{k^2+1}$$

$$\rho = \lim_{k \to \infty} \frac{k+1}{(k+1)^2+1} \times \frac{k^2+1}{k}$$

$$= \lim_{k \to \infty} \frac{\cancel{k}(1+1/k)}{k^2+2k+2} \; \frac{k^2+1}{\cancel{k}}$$

$$= 1$$ Test fail

The root test

Let Σu_k be a series with positive terms and suppose that $\rho = \lim_{k \to \infty} \sqrt[k]{u_k} = \lim_{k \to \infty} (u_k)^{1/k}$

(a) If $\rho < 1$, the series converges

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges

(c) If $\rho = 1$, the series may converges or diverge
so that another ~~test~~ test must be tried.

[Try this test when u_k involve k^{th} power)

* use the root test to determine whether the series converges. ~~If the test~~

* (1) $\sum_{k=1}^{\infty} \left(\frac{k}{100}\right)^k$

(34)

$$\rho = \lim_{k\to\infty} \left[\left(\frac{k}{100}\right)^k\right]^{1/k}$$

$$= \lim_{k\to\infty} \frac{k}{100}$$

$$= \infty.$$

$\therefore$ The series is diverges by Root test

(2) $$\sum_{k=1}^{\infty} \left(\frac{3k+2}{2k-1}\right)^k$$

$$\rho = \lim_{k\to\infty} \left[\left(\frac{3k+2}{2k-1}\right)^k\right]^{1/k}$$

$$= \lim_{k\to\infty} \frac{3k+2}{2k-1} = \lim_{k\to\infty} \frac{k\left(3+\frac{2}{k}\right)}{k\left(2-1/k\right)} =$$

$\therefore$ the series is diverges.

(3) $$\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$$

$$\rho = \lim_{k\to\infty} \left[\frac{1}{(\ln(k+1))^k}\right]^{1/k}$$

$$= \lim_{k\to\infty} \frac{1}{\ln(k+1)}$$

$= 0 < 1$, the series converges

(4) $$\sum_{k=1}^{\infty} (1-e^{-k})^k$$

$$\rho = \lim_{k\to\infty} \left((1-e^{-k})^k\right)^{1/k} = \lim_{k\to\infty} 1-e^{-k}$$

$= 1$, inconclusive

Find the general term of the series and use the ratio test to show that the series converges

(19)

(1) $1+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\cdots\cdots$

General term is, $\frac{1\cdot 2\cdot 3\cdot 4\cdots\cdots k}{1\cdot 3\cdot 5\cdot 7\cdots(2k-1)}$

$$\sum_{k=1}^{\infty}\frac{1\cdot 2\cdots(k)}{1\cdot 3\cdot 5\cdots(2k-1)}=\sum_{k=1}^{\infty}\frac{k!}{1\cdot 3\cdots(2k-1)}\cdot\frac{2\cdot 4\cdot 6\cdot 8\cdots 2k}{2\cdot 4\cdot 6\cdot 8\cdots 2k}$$

$$=\sum_{k=1}^{\infty}\frac{k!\,2\cdot 4\cdot 6\cdot 8\cdots 2k}{\cancel{2k}\,1\cdot 2\cdot 3\cdot 4\cdots(2k-1)2k}$$

$$=\sum_{k=1}^{\infty}\frac{k!\,2\cdot 4\cdot 6\cdot 8\cdots 2k}{(2k)!}$$

$$=\sum_{k=1}^{\infty}\frac{k!\,2^k\left[1\cdot 2\cdot 3\cdots k\right]}{(2k)!}$$

$$=\sum_{k=1}^{\infty}\frac{(k!)^2\,2^k}{(2k)!}$$

$\cancel{n!=n(n-1)!}$

$n!=n(n-1)!$

$$\rho=\lim_{k\to\infty}\frac{a_{k+1}}{a_k}$$

$$=\lim_{k\to\infty}\frac{((k+1)!)^2\,2^{k+1}}{(2(k+1))!}\times\frac{2k!}{(k!)^2\,2^k}$$

$$=\lim_{k\to\infty}\cancel{2}\cdot\left(\frac{(k+1)!}{k!}\right)^2\cdot\frac{2k!}{(2k+2)!}\quad 2$$

$$=\lim_{k\to\infty}2(k+1)^2\frac{2k!}{(2k+2)(2k+1)2k!}\quad 2$$

$$\lim_{K\to\infty} \frac{2(K+1)^2}{(2K+2)(2K+1)} = \lim_{K\to\infty} \frac{2(K^2+2K+1)}{4K^2+6K+2}$$

$$= 2 \times \frac{1}{4}$$

$$= \frac{1}{2} < 1 \quad \text{converges}$$

(2)

use any method to determine whether the series converges.

(1) $\sum_{K=1}^{\infty} \frac{7\cos^2 K}{K!}$

we have $\cos^2 K \leq 1$

$$\therefore \frac{7\cos^2 K}{K!} \leq \frac{7}{K!}$$

consider the series $\cancel{\sum_{K=1}^{\infty} \frac{7\cos^2 K}{K!}}$ $= \sum_{K=1}^{\infty} \frac{7}{K!} = \sum_{K=1}^{\infty} b_K$

$$\rho = \lim_{K\to\infty} \frac{b_{K+1}}{b_K} = \lim_{K\to\infty} \frac{7}{(K+1)!} \cdot \frac{K!}{7}$$

$$= \lim_{K\to\infty} \frac{1}{K+1}$$

$$= 0 < 1$$

$\sum b_K$ converges by ratio test

Hence by comparison Test $\sum a_K = \sum_{K=1}^{\infty} \frac{7\cos^2 K}{K!}$ also conv

* $\sum_{K=1}^{\infty} K^{50} e^{-K}$

$$\rho = \lim_{K\to\infty} \frac{a_{K+1}}{a_K}$$

$$= \lim_{K\to\infty} \frac{(K+1)^{50} e^{-(K+1)}}{K^{50} e^{-K}} = \lim_{K\to\infty} \left(\frac{K+1}{K}\right)^{50} e^{-1}$$

$$= \lim_{K\to\infty} \left(\frac{K(1+1/K)}{K}\right)^{50}$$

$$\frac{1}{e} < 1$$

(37)

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By ratio test the series is convergent.

* $\sum_{K=1}^{\infty} \left(\frac{[\pi(K+1)]}{K^{K+1}}\right)^{K}$

$\rho = \lim_{K\to\infty} (a_K)^{1/K}$

$= \lim_{K\to\infty} \left(\left(\frac{\pi(K+1)}{K}\right)^{K} \right)^{1/K}$

$= \lim_{K\to\infty} \frac{\pi(K+1)}{K} \frac{1}{K^{1/K}}$ $\qquad \lim_{K\to\infty} K^{1/K} = 1$

$= \pi > 1 \quad \therefore$ Divergent Series

* $\lim_{K\to 1} \sum_{K=1}^{\infty} \frac{1}{5K^2-2K}$

$K^2 \geq K$

$-K^2 \leq -K$

$-2K^2 \leq -2K$

$5K^2 - 2K^2 \leq 5K^2 - 2K$

$3K^2 \leq 5K^2 - 2K$

$\frac{1}{3K^2} \geq \frac{1}{5K^2-2K}$

$\sum_{K=1}^{\infty} \frac{1}{3K^2}$ cgt

$\therefore \sum_{K=1}^{\infty} \frac{1}{5K^2-2K}$ cgt

* $\sum_{K=1}^{\infty} \frac{\tan^{-1} K}{K^2}$

$b_K = \frac{1}{K^2}$. $\qquad \sum_{K=1}^{\infty} \frac{1}{K^2}$ is cgt

$\lim_{K\to\infty} \frac{\tan^{-1} K}{K^2} \times K^2$

$\lim_{K\to\infty} \tan^{-1} K = \tan^{-1}\infty = \pi/2$ finite . cgt

$$\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$$

$2k^2+1 \geq 2k^2$

$2k^{8/3}-1 \leq 2k^{8/3}$

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$$\frac{2k^2+1}{2k^{8/3}-1} \geq \frac{2k^2}{2k^{8/3}}$$

$$\frac{2k^2+1}{2k^{8/3}-1} \geq \frac{1}{k^{2/3}}$$ is divergent

Hence $\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$ is divergent

<u>Note</u>

Let $\sum a_k$ and $\sum b_k$ be series with +ve terms.

(a) If $\lim_{k\to\infty} (a_k/b_k) = 0$ and $\sum b_k$ converges, then $\sum a_k$

(b) If $\lim_{k\to\infty} (a_k/b_k) = \infty$ and $\sum b_k$ diverges, then $\sum a_k$

$$\sum_{k=1}^{\infty} \frac{\ln k}{k} \qquad a_n = \frac{\ln k}{k}$$

let $b_k = \frac{1}{k}$

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$ which is dgt $(P=1)$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\ln k}{k}}{1/k} = \lim_{k\to\infty} \frac{\ln k}{k} \times \frac{k}{1}$$

$$= \lim_{k\to\infty} \ln k = \underline{\underline{\infty}}$$

$\therefore \sum b_k$ diverges, then $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is dgt.

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$2-\frac{8}{3}$
$-2/3$

$$\sum_{k=1}^{\infty} \frac{2k^2+1}{2k^{8/3}-1}$$

$$a_k = \frac{2k^2+1}{2k^{8/3}-1} = \frac{2k^2\left[1+\frac{1}{2k^2}\right]}{2k^{8/3}\left[1-\frac{1}{2k^{8/3}}\right]} = \frac{1}{k^{2/3}}\frac{\left[1+\frac{1}{2k^2}\right]}{\left[1-\frac{1}{2k^{8/3}}\right]}$$

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$$b_k = \frac{1}{k^{2/3}}$$

$\sum_{k=1}^{\infty} b_k$ dgs by p series. Since $p = 2/3 < 1$

$$\therefore \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{1/k^{2/3}\left[1+1/2k^2\right]}{1/k^{2/3}\left[1-1/2k^{8/3}\right]}$$

$= 1$ finite and >0

Hence the given series is ~~convergent~~ divergent by limit Comparison test.

Alternating Series

A series in which ~~those~~ the terms are alternate positive and negative is called an Alternating Series

Eg $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$\sum_{k=1}^{\infty} (-1)^{k}\frac{1}{k} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$

In general, an alternating series has one of the following two forms;

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

$$\sum_{k=1}^{\infty} (-1)^{k} a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

24 where a_k's are assumed to be positive in both case

Absolute Convergence

A Series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$ is

said to converge absolutely if the series of absolut

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + |u_3| + \cdots + |u_k| + \cdots$$

converges and is said to diverge absolutely if the series of absolute values diverges.

Eg:- * Determine whether the following series converge absolutely.

(a) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} \cdots\cdots$

$\sum_{k=1}^{\infty} |u_k| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ which is a

~~cgt~~ Convergent geometric series. Hence the given series is convergent absolutely.

* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$$\sum_{K=1}^{\infty} |u_K| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is harmonic series. The absolute value is a divergent Harmonic series. Hence it is diverges absolutely.

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Theorem

If the series $\sum_{K=1}^{\infty} |u_K| = |u_1| + |u_2| + |u_3| + \cdots + |u_K| + \cdots$

converges, then so does the series.

$$\sum_{K=1}^{\infty} u_K = u_1 + u_2 + u_3 + \cdots + u_K + \cdots$$

Conditional convergence

An infinite series Σa_n is convergent conditionally if Σa_n is convergent but its absolute ~~value~~ series $|\Sigma a_n|$ is divergent.

Eg:- Consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots (-1)^{K+1}\frac{1}{K} + \cdots \longrightarrow (1)$$

which is a Conditionally convergent series. Because its absolute value is the divergent Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{K} + \cdots \rightarrow (2)$$

However, series (1) converges, since it is the alternating

Harmonic series and Series (2) diverges, since it is constant times the divergent Harmonic Series.
Thus (1) is a conditionally convergent series.

(26) Problems

(1) Determine whether the Series converges absolutely, converges conditionally.

$$\sum_{K=1}^{\infty} \frac{\cos K}{K^2}$$

We have $|\cos K| \leq 1$

$$\frac{|\cos K|}{K^2} \leq \frac{1}{K^2}.$$

But $\sum \frac{1}{K^2}$ is a convergent p series ($p=2$), so the Series of ~~absolute~~ absolute values converge by the Comparison test. Thus the given Series converges absolutely and hence converges.

(2) $$\sum_{K=1}^{\infty} (-1)^{K+1} \frac{K+3}{K(K+1)}$$

Given Series is ~~abo~~ absolutely convergent if,

$$\sum_{K=1}^{\infty} \left| (-1)^{K+1} \frac{K+3}{K(K+1)} \right|$$ is convergent

$$\left|\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}\right| = \sum_{k=1}^{\infty}\frac{k+3}{k(k+1)} \rightarrow (1)$$

Let $b_k = \frac{1}{k}$, then $\sum b_k = \sum \frac{1}{k}$ which is a divergent series with $p = 1$. and $a_k = \sum_{k=1}^{\infty}\frac{k+3}{k(k+1)}$

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$$\rho = \lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{\frac{k+3}{\cancel{k}(k+1)}}{\cancel{k}}$$

$$= \lim_{k\to\infty}\frac{\cancel{k}\left(1+\frac{3}{k}\right)}{\cancel{k}\left(1+\frac{1}{k}\right)} = 1$$

Here ρ is finite and $\rho > 0$. Hence $\sum_{k=1}^{\infty}\frac{k+3}{k(k+1)}$ is ~~eg~~ convergent or divergent. Since $\sum b_k$ is divergence,

$\sum_{k=1}^{\infty}\frac{k+3}{k(k+1)}$ is divergent.

From (1) $\sum_{k=1}^{\infty}|a_k|$ is divergent.

or $\sum a_k$ is absolutely divergent.

Ratio Test for absolute Convergence

Let $\sum u_k$ be a series with non zero terms and Suppose that $\rho = \lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|}$

(a) If $\rho < 1$ then the series $\sum u_k$ converges absolutely and $\therefore$ converges.

(b) If $\rho > 1$ or $\rho = \infty$ then the series $\sum u_k$ diverges

(c) If $\rho = 1$, no conclusion about convergence.

Problems Determine whether the series is convergent or divergent. (54)

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k 2^k}{k!}$

Taking the absolute value of the general term u_k, we obtain

$$|u_k| = \left|\frac{(-1)^k 2^k}{k!}\right| = \frac{2^k}{k!}$$

Thus

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \times \frac{k!}{2^k}$$

$$= 2 \lim_{k\to\infty} \frac{1}{k+1}$$

$$= 0 < 1$$

Since $\rho < 1$ which implies that the series converges hence absolutely converges. And therefore converges.

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!}{3^k}$

$$|u_k| = \left|\frac{(-1)^k (2k-1)!}{3^k}\right| = \frac{(2k-1)!}{3^k}$$

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{[2(k+1)-1]!}{3^{k+1}} \times \frac{3^k}{(2k-1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+1)!}{3} \cdot \frac{1}{(2k-1)!}$$

$$= \lim_{k\to\infty} \frac{(2k+1)\,2k\,\cancel{(2k-1)!}}{3} \cdot \frac{1}{\cancel{(2k-1)!}}$$

$$= \frac{1}{3} \lim_{K\to\infty} 2K(2K+1) = \infty$$

Which implies that the series diverges.

(c) $\sum_{K=1}^{\infty} \frac{(-1)^K K^5}{e^K}$

(29)

$$|u_K| = \left| \cancel{\sum_{K=1}^{\infty}} (-1)^K \frac{K^5}{e^K} \right| = \frac{K^5}{e^K}$$

$$\text{Thus } \rho = \lim_{K\to\infty} \frac{|u_{K+1}|}{|u_K|} = \lim_{K\to\infty} \frac{(K+1)^5}{e^{K+1}} \times \frac{e^K}{K^5}$$

$$= \lim_{K\to\infty} \frac{K^5 (1+\frac{1}{K})^5 \times}{K^5 e}$$

$$= \frac{1}{e} \lim_{K\to\infty} \left(1+\frac{1}{K}\right)^5$$

$$= \frac{1}{e} < 1$$

Since $\rho < 1$ which implies that the series converges hence absolutely converges. And therefore converges.

(d) $\sum_{K=1}^{\infty} \frac{K \cos K\pi}{K^2+1}$

We have $|\cos K\pi| = |(-1)^K| = 1$

$$\cancel{\left| \sum_{K=1}^{\infty} \frac{K \cos K\pi}{K^2+1} \right| = \sum_{K=}}$$

$$\left| \sum_{K=1}^{\infty} \frac{K \cos K\pi}{K^2+1} \right| = \sum_{K=1}^{\infty} \frac{K}{K^2+1} \to (1) \quad \text{Let } a_K = \frac{K}{K^2+1}$$

choose $b_k = \frac{1}{k}$

Now $\sum b_k$ is divergent with p series $p = 1$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k}{k^2+1}}{1/k}$$

$$= \lim_{k\to\infty} \frac{k}{k^2+1} \times k$$

$$= \lim_{k\to\infty} \frac{k}{\cancel{k^2}\left(1+\frac{1}{k^2}\right)} \times k^2$$

$$= \lim_{k\to\infty} \frac{\cancel{k}}{\cancel{k}\left(1+\frac{1}{k^2}\right)}$$

$$= \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \text{ finite}$$

Hence the given series is cgt or divergent together. Since b_k is divergent

$$\therefore \sum_{k=1}^{\infty} \left| \frac{k \cos k\pi}{k^2+1} \right| \text{ is divergent}$$

$\therefore$ Given series is not absolutely convergent.

H.W ① $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3^k}{k^2}$

$$|u_k| = \left| (-1)^{k+1} \frac{3^k}{k^2} \right| = \frac{3^k}{k^2}$$

Thus $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{3^{k+1}}{(k+1)^2} \times \frac{k^2}{3^k}$

$$= 3 \lim_{k\to\infty} \frac{k^2}{k^2\left[1+\frac{1}{k^2}\right]} = 3 \geq 1,$$ wh

ρ finite, > 0

Thus $\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{3^{(k+1)}}{(k+1)^2} \times \frac{k^2}{3^k}$

$$= \lim_{k\to\infty} \frac{3\,\cancel{k^2}}{\cancel{k^2}\left(1+\frac{1}{k^2}\right)}$$

(31)

$= 3 > 1$

∴ Thus the given series is divergent. Hence not absolutely convergent.

<u>Leibnitz's</u> <u>Test</u> on <u>Alternating</u> <u>Series</u>

The alternating series $\sum(-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 \ldots$

cgs if (i) $u_n > u_{n+1}$ $\forall n$ and (ii) $\lim_{n\to\infty} u_n = 0$.

<u>Probs</u>

1) Examine the convergence of the series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$

$u_n = \frac{1}{n}$ $\quad u_{n+1} = \frac{1}{n+1}$ $\quad$ (1) $u_n > u_{n+1}$ $\forall n$.

(2) $\lim_{n\to\infty} u_n = 0$

∴ By Leibnitz's test the series cgt

2) Examine the cgence of the series

$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} \ldots$

$u_n = \frac{n+1}{n}$ $\quad u_{n+1} = \frac{n+2}{n+1}$.

(1) $u_n - u_{n+1} = \frac{n+1}{n} - \frac{n+2}{n+1} = \frac{(n+1)^2 - n(n+2)}{n(n+1)} = \frac{1}{n(n+1)} > 0$ $\forall n$

3) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - -$

$$u_n = \frac{1}{(n+1)^3}\left[1+2+--+n\right] = \frac{n(n+1)}{2(n+1)^3}$$

$$= \frac{n}{2(n+1)^2}$$

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$$u_{n+1} = \frac{n+1}{2(n+2)^2}.$$

$$u_n - u_{n+1} = \frac{n}{2(n+1)^2} - \frac{n+1}{2(n+2)^2}$$

$$= \frac{n(n+2)^2 - (n+1)(n+1)^2}{2(n+1)^2(n+2)^2}$$

$$= \frac{n[n^2+4n+4] - (n+1)[n^2+2\ldots}{2(n+1)^2(n+2)^2}$$

$$= \frac{n^2+n-1}{2(n+1)^2(n+2)^2} > 0 \quad \forall n$$

$u_n > u_{n+1} \quad \forall n$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2} = 0$

$\therefore$ By Leibnitz's Test Series cgs

(1)

Multivareable Calculus - Differenciation

Parteal derevatives

If f is a function of one vareable then the derevative of f w.r.to x is denoted by $\frac{df}{dx}$.

If 'f' is a function of two vareables x & y then the derevatives are called parteal derevatives and partial derevative of f w.r.to x is denoted by $\frac{\partial f}{\partial x}$ or f_x.

partial derivative of f w.r.to y is denoted by $\frac{\partial f}{\partial y}$ or f_y.

Problems

1. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$

-> A

$$Z = x^4 \sin(xy^3).$$

$$\frac{\partial z}{\partial x} = x^4 \cos(xy^3) \cdot y^3 + \sin(xy^3) \times 4x^3$$

$$\frac{\partial z}{\partial y} = x^4 \cos(xy^3) \times 3xy^2.$$

2. $f(x,y) = 2x^3y^2 + 2y + 4x$ find $f_x(1,3)$ & $f_y(1,3)$

-> $f_x = 6x^2y^2 + 4$ $\quad f_x(1,3) = 58$

3 $f(x,y,z) = x^3y^2z^4 + 2xy + z$ Compute f_x, f_y, f_z

→ $f_x = y^2z^4 + 2y$ $f_y = 2yx^3z^4 + 2x$ $f_z = 4x^3y^2z^3 + 1$

4 $f(\rho, \phi, \theta) = \rho^q \cos\phi \sin\theta$. Find f_ρ, f_θ, f_ϕ

5 $z = e^{3x} \sin y$ Find $\frac{\partial z}{\partial x}$ at $(x,0)$ and $\frac{\partial z}{\partial y}(\log 3, 0)$

6 $f(x,y) = xe^{-y} + 5y$. Find the slope of the surface $z = f(x,y)$ in the x-direction at $(2,5)$

→ Slope of z in the x direction $= \frac{\partial z}{\partial x}$

$= e^{-y}$

at $(2,5$ $= \underline{\underline{e^{-5}}}$

7 $f(x,y) = \sin(y^2 - 4x)$ find the rate of change of the surface $z = f(x,y)$ w.r.to y at the pt $(3,1)$ with x fixed

→ $\frac{\partial f}{\partial y} = \cos(y^2 - 4x) \times 2y$

at $(3,1)$ $= \cos(1 - 4\times 3) \times 2$ $= 2\cos(-11)$ $= 2\cos 11$

8. $z = (x+y)^{-1}$ find $\frac{\partial z}{\partial x}$ at $(-1,4)$

$\frac{\partial z}{\partial x} = -\frac{1}{(x+y)^2}$ at $(-1,4)$ $= -1/9$

9. A pt moves along intersection of any thick paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$ at what rate is z changing w.r.to x when the pt at $(3,1,12)$

→ Given $z = x^2 + 3y^2$ and $y = 1$ ...

$\frac{\partial z}{\partial x} = 2x$ at $(3,1,12) = 2\times 3 = 6.$ (2)

10 $f(x,y,z) = x^2y^4z^3 + xy + z^2+1$ find f_x, f_y, and f_z (1,2,3)

→ $f_x(1,2,3) = 866$ $f_y(1,2,3) = 865$ $f_z(1,2,3) = 438$

11 If $f(x,y) = y^2e^x + y$ find f_{xyy}

→ $f_x = y^2e^x.$

$f_{xy} = 2ye^x$

$f_{xyy} = 2e^x.$

12. $f(x,y) = y^3e^{-5x}$ find f_{yyxx} at $(0,1)$

$f_y = 3y^2e^{-5x}.$

$f_{yy} = 6y\,e^{-5x}$

$f_{yyx} = -30y\,e^{-5x}$

$f_{yyxx} = 150y\,e^{-5x}.$

$f_{yyxx}(0,1) = 150$

Higher Order Partial derivatives

$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$

$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}.$ $\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$

The last two partial derivatives are called mixed partial derivatives.

Differentiability

A function f of two variables x, y is said to be differentiable at (x_0, y_0) Provided $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exists and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

where $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$

A function f of three variables x, y, z is said to be differentiable at (x_0, y_0, z_0) if $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$, $f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z) \to (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} = 0$$

Where $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$

Problems

1 S.T $f(x,y) = x^2 + y^2$ is differentiable at $(0,0)$

$\to$ $f_x = 2x$ $\qquad f_y = 2y$

$f_x(0,0) = 0$ $\qquad f_y(0,0) = 0$

$\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0,0) = \Delta x^2 + \Delta y^2$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta x^2 + \Delta y^2}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sqrt{\Delta x^2 + \Delta y^2} = 0.$$

2) S.T $f(x,y,z) = x^2+y^2+z^2$ is differentiable at $(0,0,0)$

Theorem

1) If a function is differentiable at a point it is continuous at that point.

2) If all 1st order partial derivatives exist and are continuous at a point then f is differentiable at that point.

Pbms

3) S.T $f(x,y,z) = x+yz$ differentiable every where

→ $\frac{\partial f}{\partial x} = 1$ $\frac{\partial f}{\partial y} = z$ $\frac{\partial f}{\partial z} = y$ are defined and continuous every where. So f is diff everywhere.

4) S.T $f(x,y) = x^2 \sin y$ is differentiable everywhere.

5) S.T $f(x,y,z) = xy \sin z$ is differentiable every where.

Differentials

If $z = f(x,y)$ is differentiable at a point (x,y) then $\boxed{dz = f_x(x,y)dx + f_y(x,y)dy}$ is the total differential of z or f at (x,y).

If $w = f(x,y,z)$ then

$\boxed{dw = f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz}$ is called total differential of w at (x,y,z).

Change Δz in $z \simeq dz$

ie Change Δz in z is approximately the differential dz where dx is change in x and dy is change in y.

If $\Delta x, \Delta y$ are close to 0, the magnitude of the error in the approximation will be much smaller than the distance $\sqrt{\Delta x^2 + \Delta y^2}$ b/w (x, y) and $(x+\Delta x, y+\Delta y)$

Problems

1) Find approximately the change in $z = xy^2$ at $(0.5, 1)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in the approximation with the distance b/w $(0.5, 1)$, $(0.503, 1.004)$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= y^2 dx + 2xy\, dy$$

$dx = .003$ $\quad dy = .004$

$\therefore dz = 1 \times .003 + 2 \times .5 \times 1 \times .004 = .007$

Change Δz in z is $.007$

By actual calculation change Δz in z is

$.503(1.004)^2 - .5 \times (1)^2 = \underline{.007032048}$

Error $= .0000 32048$.

Distance b/w pts $= \sqrt{(.003)^2+(.004)^2} = .005$ (4)

$\therefore \dfrac{|dz-\Delta z|}{\sqrt{\Delta x^2+\Delta y^2}} = \dfrac{.00003 2048}{.005} = .0064096 < 1/150.$

2 The lengths, width and height of a rectangular box are measured with an error atmost 5%. Find the maximum % error that result if these quantities are used to calculate the diagonal of the box.

→ If x is length, y breadth, z height.

then $D = \sqrt{x^2+y^2+z^2}$

differential $dD = \dfrac{\partial D}{\partial x}dx + \dfrac{\partial D}{\partial y}dy + \dfrac{\partial D}{\partial z}dz$

$= \dfrac{1}{2\sqrt{x^2+y^2+z^2}} 2x\,dx + \dfrac{1}{2\sqrt{x^2+y^2+z^2}} 2y\,dy + \dfrac{1}{2\sqrt{x^2+y^2+z^2}} 2z\,dz$

$$dD = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$$

Given $\left|\dfrac{\Delta x}{x}\right| \le 0.05$ $\left|\dfrac{\Delta y}{y}\right| \le 0.05$ & $\left|\dfrac{\Delta z}{z}\right| \le 0.05$

$$\therefore \frac{\Delta D}{D} \approx \frac{dD}{D} = \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\Delta x + y\Delta y + z\Delta z}{x^2+y^2+z^2}$$

$$= \frac{x^2\cdot\frac{\Delta x}{x} + y^2\cdot\frac{\Delta y}{y} + z^2\cdot\frac{\Delta z}{z}}{x^2+y^2+z^2}$$

$$\leq \frac{0.05\,(x^2+y^2+z^2)}{x^2+y^2+z^2} = 0.05$$

$\therefore$ Max % of error in D is 5%.

Local linear approximation

f is a differentiable at a point (x_0, y_0)

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

is called local linear approximation to f at (x_0, y_0).

If f is function of three variables and f is differentiable at (x_0, y_0, z_0) then local linear approximation to f at (x_0, y_0, z_0) is

$$L(x,y,z) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$$

Pbms

1 Let $L(x,y)$ denote the local linear approximation to $f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$. Compare the error in approximating $f(3.04, 3.98)$ by $L(3.04, 3.98)$ with the distance b/w $(3,4)$, $(3.04, 3.98)$

$\Rightarrow$ $L(x,y) = f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4)$

$= 5 + \frac{3}{5}(x-3) + 4/5\,(y-4)$

$L(3.04, 3.98) = 5 + 3/5 \times .04 + 4/5 \times -0.02 = 5.008$

$f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.00819.$

Error $= .00019.$

Distance b/w the pts $\approx \sqrt{(.04)^2 + (.02)^2} \approx .045$

Error less than 1/200 of the distance b/w the pts.

---

2. Find Local linear approximation $f(x,y,z) = xyz$ at the pt $P(1,2,3)$. Compare the error in approximating f by L at the specified pt $Q(1.001, 2.002, 3.003)$ with the distance b/w P and Q.

$\Rightarrow$ $L(x,y,z) = 6 + 6(x-1) + 3(x-2) + 2(x-3)$

$L(1.001, 2.002, 3.003) = 6.018$

$f(1.001, 2.002, 3.003) = 6.018018006.$

Error $= .000018$

Distance $= .00374165$

Error < 1/200 of distance b/w the pts

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3. Find Local linear approximation L to function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ at (4,3). Compare the error in approximating f by L at the

Chain Rule

If $x = x(t)$ and $y = y(t)$ are differentiable at t and if $z = f(x,y)$ is differentiable at the pt $(x,y) = (x(t), y(t))$ then z is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

If $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable at t and $w = f(x,y,z)$ is differentiable at t

and $$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

Problems

1. If $x = t^2$, $y = t^3$ where $z = x^2 y$ find $\frac{dz}{dt}$

$\rightarrow$ $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2xy \cdot 2t + x^2 \cdot 3t^2$$

$$= 2 \times t^2 \times t^3 \times 2t + (t^2)^2 \times 3t^2 = 4t^6 + 3t^6 = \underline{7t^6}$$

2. $w = \sqrt{x^2 + y^2 + z^2}$ $\quad x = \cos\theta \quad y = \sin\theta \quad z = \tan\theta$.

find $\frac{dw}{d\theta}$ when $\theta = \pi/4$ [Ans $\sqrt{2}$]

3. $z = \log(2x^2 + y)$ $\quad x = \sqrt{t} \quad y = t^{1/3}$ find dz/dt

Chain rule for Partial differentiation

If $x = x(u,v)$, $y = y(u,v)$ have 1st order partial derivatives at (u,v) and if z is differentiable at (x,y) then z has first order partial derivatives at (u,v) given by

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Problems

1 Given $z = e^{xy}$ $\quad x = 2u+v \quad y = \frac{u}{v}$ find.

$\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = ye^{xy} \cdot 2 + xe^{xy} \cdot \frac{1}{v}$$

$$= \frac{2u}{v} e^{(2u+v)\frac{u}{v}} + \frac{(2u+v)}{v} e^{(2u+v)\frac{u}{v}}.$$

$$= e^{(2u+v)\frac{u}{v}} \left[\frac{4u}{v} + 1\right]$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = ye^{xy} \cdot 1 + xe^{xy} \cdot \frac{-u}{v^2}$$

$$= e^{xy}\left[y - \frac{ux}{v^2}\right]$$

$$= e^{(2u+v)\frac{u}{v}} \left[\frac{u}{v} - \frac{u(2u+v)}{v^2}\right].$$

$$= e^{(2u+v)\frac{u}{v}} \left[-\frac{2u^2}{v^2}\right]$$

2. $w = e^{xyz}$ $x = 3u+v$ $y = 3u - v$ $z = u^2 v$

find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$

→

$$\frac{\partial w}{\partial u} = e^{xyz}\left[3yz + 3xz + 2xyuv\right]$$

$$\frac{\partial w}{\partial v} = e^{xyz}\left[yz - xz + xy \cdot u^2\right]$$

3. if $w = x^2 + y^2 - z^2$

$x = \rho \sin\phi \cos\theta$ $y = \rho \sin\phi \sin\theta$ $z = \rho \cos\phi$

find $\frac{\partial w}{\partial \rho}$ and $\frac{\partial w}{\partial \theta}$.

→ $-2\rho \cos 2\phi$, 0.

4. $w = xy + yz$ $y = \sin x$ $z = e^x$.

find $\frac{dw}{dx}$

→ $x \cdot \sin x + e^x \sin x$.

5. $z = 3x^2 y^3$ $x = t^4$ $y = t^3$ find $\frac{dz}{dt}$

6. $z = \sqrt{1 + x - 2xy^4}$ $x = \log t$, $y = 2t$ find $\frac{dz}{dt}$

7. $z = 8x^2 y - 2x + 3y$ $x = uv$ $y = u + v$ find $\frac{\partial z}{\partial u}$ & $\frac{\partial z}{\partial v}$

8. $w = 5\cos(xy) - \sin(xz)$ $x = 1/t$ $y = t$ $z = t$ find $\frac{dw}{dt}$

9. find $\frac{\partial f}{\partial u}$ at $u = 1, v = -2$ and $\frac{\partial f}{\partial v}$ at $u = 1, v = -2$

Where $f = x^2 y^2 - x + 2y$, $x = \sqrt{u}$, $y = uv^3$.

Theorem

If the equation $f(x,y)=c$ defines implicitly as differential function of x then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \text{if } \frac{\partial f}{\partial y} \neq 0 .$$

Pbm

Given $x^3+y^2x-3=0$ find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{\partial f}{\partial x} = 3x^2+y^2$$

$$\frac{\partial f}{\partial y} = 2xy \qquad \frac{dy}{dx} = -\frac{(3x^2+y^2)}{2xy}.$$

Theorem

If $f(x,y,z)=c$ define z implicitly as a differentiable function of x,y and if $\frac{\partial f}{\partial z} \neq 0$

$$\left.\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} \right\} \text{ if } \frac{\partial f}{\partial z} \neq 0.$$

Pbm Given $x^2+y^2+z^2=1$ then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

at $(2/3, 1/3, 2/3)$

$$\frac{\partial z}{\partial x} = -x/z \qquad \frac{\partial z}{\partial y} = -y/z$$

at the pt. $\frac{\partial z}{\partial x} = -1 \qquad \frac{\partial z}{\partial y} = -1$

Maxima and Minima of functions of two variables

(i) A function f of two variables is said to have a relative maximum at (x_0, y_0) if there is a disc covered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for every points (x, y) in the disc and absolute max at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for every points (x, y) in the domain of f.

(ii) A function f of two variables is said to have a relative minimum at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for every points (x, y) in the disc and Absolute minimum if $f(x_0, y_0) \leq f(x, y)$ for every points (x, y) in the domain of f.

Note: If f has a relative maximum or relative minimum at (x_0, y_0) then we say f has a relative extremum at a point (x_0, y_0).

Theorem.

If $f(x, y)$ has a relative extremum at a point (x_0, y_0) and if the 1st order partial derivative of f exist at this point and $f_x(x_0, y_0) = 0$, $f_y(x_0, y_0) = 0$ then the point (x_0, y_0) is called a critical point

Let $p = f_x(x,y)$, $q = f_y(x,y)$

$r = f_{xx}(x,y)$ $s = f_{xy}(x,y)$ $t = f_{yy}(x,y)$

Thm

Let $D = rt - s^2$ then at a critical point (x_0, y_0)

1) If $D>0$ and $r>0$, we say that f has a relative minimum at (x_0, y_0).

(ii) If $D>0$ and $r<0$ then f has a relative maximum at (x_0, y_0)

(iii). If $D<0$ then f has a saddle point at (x_0, y_0) i.e. neither max or minimum.

(iv). If $D=0$ then no conclusion can be made.

Problems

1 Find the relative extremum of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$.

$p = f_x = 6x - 2y$ $q = f_y = -2x + 2y - 8$.

Critical points

$f_x = 0$ & $f_y = 0$

$6x - 2y = 0$ & $-2x + 2y - 8 = 0$. $\Rightarrow x = 2, y = 6$

$r = f_{xx} = 6$ $t = f_{yy} = 2$ $s = f_{xy} = -2$

$rt - s^2$ at $(2,6)$ $= 12 - 4 > 0$ $r = 6 > 0$.

f has a relative minimum at $(2,6)$

and minimum value is $f = 3(2)^2 - 2(2)(6) + 6^2 - 8\times 6$

2) Find the extremum of the function.

$f(x,y) = 4xy - x^4 - y^4$

→ $f_x = 4y - 4x^3$. $f_y = 4x - 4y^3$

Critical point $f_x = 0$, $f_y = 0$.

$4y - 4x^3 = 0$ $4x - 4y^3 = 0$.

$y = x^3$ $4x - 4x^9 = 0$

$4x(1 - x^8) = 0$

$x = 0$ $x^8 = 1$

$x = 1, -1$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$x = -1 \Rightarrow y = -1$

r s t

$f_{xx} = -12x^2$ $f_{xy} = 4$ $f_{yy} = -12y^2$.

Points	r	t	s	$D = rt - s^2$.
$(0,0)$	0	0	4	-16 → Saddle point lo
$(1,1)$	-12	-12	4	128 Relative
$(-1,-1)$	-12	-12	4.	128 Maxima

$(0,0)$ → Saddle point

Relative Maximum at $(1,1)$ & $(-1,-1)$

3) $f(x,y) = 2xy - x^3 - y^2$

→ $(0,0)$ → Saddle point, Relative maxima at $(2/3, 2/3)$

4) $f(x,y) = y^2 + xy + 4y + 2x + 3$.

5) $f(x,y) = x^2 + xy - 2y - 3x + 1$

6) $f($

Absolute extremum

Step 1: Find the critical points of f that lies in the interior of R

Step 2 Find all boundary points at which the absolute extrema can occur.

Step 3: Evaluate $f(x,y)$ at these points.

Largest of these values is absolute maximum and smallest absolute minimum.

Pbm

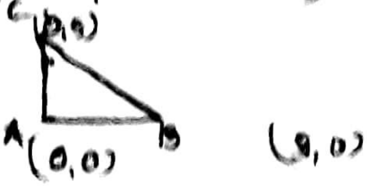
1 Find the absolute maximum and minimum of $f(x,y) = 3xy - 6x - 3y + 7$ on a closed triangular region with vertices $(0,0)$, $(3,0)$ and $(0,5)$.

Step 1 $f_x = 3y - 6$. $\quad f_x = 0 \Rightarrow y = 2$

$f_y = 3x - 3$. $\quad f_y = 0 \Rightarrow x = 1$.

Critical pt. $(1,2)$

Step 2



AB $y = 0$. $\quad f = -6x + 7$

$f_x = -6 \neq 0 \Rightarrow$ No critical pt.

AC $x = 0$. $\quad -3y + 7 \quad f_y = -3 \neq 0 \Rightarrow$ No critical pt

BC $\dfrac{y-0}{5-0} = \dfrac{x-3}{0-3}$ $\quad y = -\dfrac{5}{3}x + 5$

$$f(x) = 3x\left(-\frac{5}{3}x+5\right) - 6x - 3\left[-5/_3x+5\right] + 7$$

$$= -5x^2 + 15x - 6x + 5x - 15 + 7$$

$$= -5x^2 + 14x - 8$$

$f_x = 0 \Rightarrow -10x + 14 = 0 \quad x = 7/5$

$\Rightarrow y = -5/_3 \times 7/5 + 5 = 8/3$

Critical pt $(7/5, 8/3)$

Step 3

(x,y)	$(1,2)$	$(7/5, 8/3)$	$(0,0)$	$(3,0)$	$(0,5)$
$f(x,y)$	1	$9/5$	7	-11	-8

Absolute maxima at $(0,0)$, and absolute minima at $(3,0)$

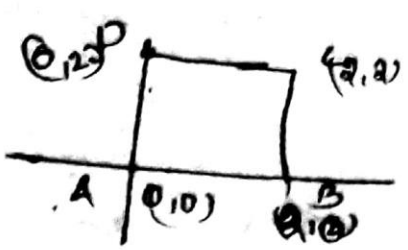
---

2) $f(x,y) = x^2 - 3y^2 - 2x + 6y$ where R is the region bounded by the square with vertices $(0,0)$, $(0,2)$, $(2,2)$, and $(2,0)$

$\rightarrow$ $f_x = 2x - 2 \qquad f_x = 0 \Rightarrow x = 1$

$f_y = -6y + 6. \qquad f_y = 0 \Rightarrow y = 1.$

Critical point $(1,1)$.



AD $x=0$.

$f(y) = -3y^2+6y$ $\quad f_y = -6y+6$ $\quad f_y=0 \Rightarrow y=1$.

Critical pt $(0,1)$.

BC $x=2$ $\quad f(y) = 4-3y^2-4+6y$

$f_y = 0 \Rightarrow -6y+6 \Rightarrow y=1$

Critical point $(2,1)$

AB $y=0$ $\quad f(x) = x^2-2x$

$f_x = 0 \Rightarrow 2x-2=0 \quad x=1$

pt $(1,0)$.

DC $y=2$ $\quad f(x) = x^2-12-2x+12$

$f_x = 0 \Rightarrow 2x-2=0 \quad x=1$

pt $(1,2)$

(x,y)	$(0,0)$	$(2,0)$	$(2,2)$	$(0,2)$	$(1,1)$	$(0,1)$	$(2,1)$	$(1,0)$	$(1,2)$
$f(x,y)$	0	0	0	0	2.	3	3	-1	-1

Absolute Maximum at $(0,1)$ & $(2,1)$

Absolute Minimu at $(1,0)$ & $(1,2)$

Derivatives

If $r(t)$ is a vector valued function, we define the derivatives of 'r' with respect to 't' to be the vector valued function r' given by

$$r'(t) = \lim_{h \to 0} \frac{r(t+h) - r(t)}{h}.$$

The domain r' consists of all values of t in the domain of $r(t)$ for which the limit exists.

The derivatives of $r(t)$ can be expressed as

$$\frac{d}{dt}[r(t)], \quad \frac{dr}{dt}, \quad r'(t), \quad r'$$

Eg: Let $r(t) = t^2 i + e^t j - 2\cos \pi t \; k$ then

$$r'(t) = 2t i + e^t j + 2\pi \sin\pi t \; \bar{k}$$

Pbms

1) find $r'(t)$ if (1) $r(t) = 6i - \sin t j$.

(2) $r(t) = \tan^{-1} t \, i + t\cos t j - 2\sqrt{t} \bar{k}$

Ans

1) $r(t) = 6i - \sin t j$

$r'(t) = -\cos t j$.

(2) $\tan^{-1} t \, i + t\cos t j - 2\sqrt{t} k$

$$r'(t) = \frac{1}{1+t^2} i + \left[t \times -\sin t + \cos t\right] j - 2\frac{1}{2\sqrt{t}} k$$

$$= \frac{1}{1+t^2} i + \left[\cos t - t\sin t\right] j - \frac{1}{\sqrt{t}} k$$

Rules of Differenciation

Let $r(t)$, $r_1(t)$, $r_2(t)$ be differentiable vector valued functions that are all in 2 space or all in 3 space and let $f(t)$ be a differentiable real valued function, k scalar and c is a constant vector. Then the following rules of differenciation holds.

(1) $\frac{d}{dt}(c) = 0$

(2) $\frac{d}{dt}[k\,r(t)] = k\frac{d}{dt}[r(t)]$

(3). $\frac{d}{dt}\left[r_1(t)+r_2(t)\right] = \frac{d}{dt}r_1(t)+\frac{d}{dt}r_2(t)$

(4) $\frac{d}{dt}\left[r_1(t)-r_2(t)\right] = \frac{d}{dt}r_1(t)-\frac{d}{dt}r_2(t)$

(5) $\frac{d}{dt}\left[f(t)r(t)\right] = f(t)\frac{d}{dt}\left[r(t)\right]+\frac{d}{dt}\left[f(t)\right]\cdot r(t)$

Geometric interpretation of Derivative.

Suppose that 'C' is the graph of a vector valued function $r(t)$ in 2-space or 3-space and that $r'(t)$ exists and is nonzero for a given value of t. If the vector $r'(t)$ is positioned with its initial point at the terminal point of the radius vector $r(t)$, then $r'(t)$ is tangent to C and points in the direction of increasing.

Pa

$h>0$

If $r(t)$ is a vector valued function, then r is differentiable at 't' if and only if each of its component functions is differentiable at t, in which case the component fns. of $r'(t)$ are the derivatives of the corresponding component fns. of $r(t)$

$r'(t) = x'(t)i + y'(t)j$.

Motion along a curve.

If $r(t)$ is the position vector of a particle moving along a curve in 2-space or 3-space, then the instantaneous velocity, instantaneous acceleration and instantaneous speed of the particle at time 't' are defined by.

$$\text{Velocity} = v(t) = \frac{dr}{dt}$$

$$\text{Acceleration} \quad a(t) = \frac{dv}{dt} = \frac{d^2r}{dt^2}$$

$$\text{Speed} = \|v(t)\| = \frac{ds}{dt}$$

	2 space	3 space,
Position	$r(t) = x(t)i + y(t)j$	$r(t) = x(t)i + y(t)j + z(t)k$
Velocity	$v(t) = \frac{dx}{dt}i + \frac{dy}{dt}j$	$v(t) = \frac{dx}{dt}i + \frac{dy}{dt}j + \frac{dz}{dt}k$
Acceleration	$a(t) = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j$	$a(t) = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j + \frac{d^2z}{dt^2}k$
Speed	$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	$\|v(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Pblms

1) A particle is moving in a helical path and its position vector at time 't' is given by $r(t) = -\cos t\, i + \sin t\, j + tk$.
Find the velocity, acceleration, speed at 6th second.

Ans

Velocity $v(t) = \frac{dr}{dt} = \sin t\, i + \cos t\, j + k$

Velocity at $t = 6$ is $\sin 6 i + \cos 6 j + k$

Acceleration $a(t) = \frac{dv}{dt} = +\cos t\, i - \sin t\, j$

at $t = 6$ $a(t) = \cos 6 i - \sin 6 j$

Speed $= \|v(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$.

2) A particle moves along a circular path in such a way that x and y co-ordinates at time 't' are $x = 2\cos t$, $y = 2\sin t$. Find velocity, acceleration and speed of the particle at time t.

Ans: $r(t) = 2\cos t\, i + 2\sin t\, j$.

Velocity $v(t) = -2\sin t\, i + 2\cos t\, j$

Acceleration $a(t) = -2\cos t\, i - 2\sin t\, j$

Speed $= \|v(t)\|$

$= \sqrt{4\sin^2 t + 4\cos^2 t}$

$= 2$.

3) A particle moves through 3-space in such a way that its velocity is $v(t) = i + tj + t^2k$.
Find the co-ordinates of the particle at time $t=1$ given that the particle is at the point $(-1, 2, 4)$ at time $t=0$.

$\rightarrow$

Given $v(t) = i + tj + t^2k$.

$$r(t) = \int v(t)\,dt = \int [i + tj + t^2k]\,dt$$

$$= ti + \frac{t^2}{2}j + \frac{t^3}{3}k + C \longrightarrow (1).$$

given $t=0$. $r = -i + 2j + 4k$

$\Longrightarrow$ $-i + 2j + 4k = C$. (Sub in (1).

$$r(t) = ti + \frac{t^2}{2}j + \frac{t^3}{3}k - i + 2j + 4k$$

$$= (t-1)i + \left(\frac{t^2}{2} + 2\right)j + \left(\frac{t^3}{3} + 4\right)k$$

at $t=1$ $r(1) = (1-1)i + (1/2 + 2)j + (1/3 + 4)k = 0i + \frac{5}{2}j + \frac{13}{3}k$

∴ The co-ordinate of the particle at $t=1$ is $(0, 5/2, 13/3)$

---

4) $r(t) = 3ti + 2t^2j + tk$

Find velocity, Acceleration & speed at 3rd second

$\rightarrow$ Velocity $v(t) = 3i + 4tj + k$. at $t=3$ $V = 3i + 12j + k$

Acceleration $a(t) = 4j$ at $t=3$ $a = 4j$

Speed $= \|v(t)\| = \sqrt{3^2 + 16t^2 + 1} = \sqrt{10 + 16t^2}$

at $t=3$ $= \sqrt{10 + 16(3)^2} = \sqrt{154}$

Gradient.

If f is a function of x and y then gradient of f is defined by $\nabla f(x,y) = f_x(x,y)i + f_y(x,y)j$

If f is a function of 3 variables x, y and z then gradient of f is defined by

$$\nabla f(x,y,z) = f_x(x,y,z)\,i + f_y(x,y,z)\,j + f_z(x,y,z)\,k$$

Del Operator $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$.

[vector Differential Operator]

Properties of Gradient

- If $\phi(x,y,z)$ is a constant function then $\nabla\phi = 0$ fun
- $\nabla(\alpha\phi) = \alpha\nabla\phi$ (α scalar)
- $\nabla(\phi \pm \psi) = \nabla\phi \pm \nabla\psi$
- $\nabla(\phi.\psi) = \phi\nabla\psi + \psi\nabla\phi$.

Pbm Find ∇z where $z = \frac{6xe^{3y}}{x+8y}$

$\rightarrow$ $\nabla z = f_x\,i + f_y\,j$

$$f_x = \frac{(x+8y)6e^{3y} - 6xe^{3y}}{(x+8y)^2} = \frac{48ye^{3y}}{(x+8y)^2}.$$

$$f_y = \frac{(x+8y)18xe^{3y} - 6xe^{3y}\times 8}{(x+8y)^2} = \frac{6xe^{3y}[3+24y-8]}{(x+3y)^2}.$$

$$\therefore \nabla z = \frac{48ye^{3y}}{(x+8y)^2}\,i + \frac{6xe^{3y}[3+24y-8]}{(x+8y)^2}\,j$$

Applications of Gradient

In medical applications the operation of certain diagnostic equipment is designed to locate heat sources generated by tumors or infections, and in military applications the trajectories of heat seeking missiles are controlled to seek and destroy enemy aircraft.

Directional derivatives

If $f(x,y)$ is differentiable at (x_0,y_0) and if $u = u_1 i + u_2 j$ is a unit vector then the directional derivative $D_u f(x_0,y_0)$ is the direction of u is given by

$$\boxed{\begin{gathered} D_u f(x_0,y_0) = f_x(x_0,y_0)\, u_1 + f_y(x_0,y_0)\, u_2 \\ \text{or} \\ D_u f = \nabla f \cdot u \end{gathered}}$$

and if $f(x,y,z)$ is differentiable at (x_0,y_0,z_0) and $u = u_1 i + u_2 j + u_3 k$ is a unit vector.

then $D_u f(x_0,y_0,z_0)$ is the direction of $\bar{u}$ is

$$\boxed{\begin{gathered} D_u f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\, u_1 + f_y(x_0,y_0,z_0)\, u_2 + f_z(x_0,y_0,z_0)\, u_3 \\ \text{or} \quad D_u f = \nabla f \cdot u \end{gathered}}$$

Pbms

1) Find $D_u f$ at P, $f(x,y) = \sin(5x-3y)$, $P(3,5)$, $u = \frac{3}{5} i - \frac{4}{5} j$.

$D_u f = f_x u_1 + f_y u_2 + f_z u_3$.

$f_x = \cos(5x-3y) \times 5$ $\quad f_x(3,5) = 5\cos(15-15) = 5$

$f_y = \cos(5x-3y) \times -3$ $\quad f_y(3,5) = -3\cos(15-15) = -3$

$u_1 = 3/5$ $\quad u_2 = -4/5$

$D_u = f_x u_1 + f_y u_2 = 5 \times 3/5 + -3 \times -4/5 = 3 + \frac{12}{5} = \frac{27}{5}$

2) Find the directional derivative of f at p in the direction of $\bar{a}$:

1) $f(x,y) = e^{x}\cos y \qquad P(0, \pi/4) \qquad a = 5i - 2j$

$\longrightarrow$ $f_x = e^{x}\cos y \qquad f_x(0,\pi/4) = 1/\sqrt{2}$

$f_y = -e^{x}\sin y \qquad f_y(0,\pi/4) = -1/\sqrt{2}$

$u = \frac{a}{\|a\|} = \frac{5i-2j}{\sqrt{5^2+(-2)^2}} = \underbrace{\frac{5}{\sqrt{29}}}_{u_1} i - \underbrace{\frac{2}{\sqrt{29}}}_{u_2} j$

$D_u = f_x u_1 + f_y u_2 = \frac{1}{\sqrt{2}} \times \frac{5}{\sqrt{29}} + \frac{-1}{\sqrt{2}} \times \frac{-2}{\sqrt{29}}$

$= \frac{5+2}{\sqrt{58}} = \frac{7}{\sqrt{58}}$

---

2) $f(x,y,z) = \frac{z-x}{z+y} \qquad P(1,0,-3) \qquad a = -6i + 3j - 2k$

$\longrightarrow$ $u = a/\|a\| = \frac{-6i+3j+2k}{\sqrt{(-6)^2+3^2+(-2)^2}} = \frac{-6}{7}i + \frac{3}{7}j - \frac{2}{7}k$

$f_x = -\frac{1}{z+y} \qquad f_x(1,0,-3) = 1/3$

$f_y = (z-x) \times -1/(z+y)^2 \qquad f_y(1,0,-3) = \frac{4}{9}$

$f_z = \frac{(z+y)-(z-x)}{(z+y)^2} = \frac{y+x}{(z+y)^2} \qquad f_z(1,0,-3) = 1/9$

$D_u = f_x u_1 + f_y u_2 + f_z u_3 = 1/3 \times -6/7 + 4/9 \times 3/7 + -2/7 \times 1/9$

$= -8/63$

---

3) Find the directional derivative of $f = \frac{x-y}{x+y}$ at $(-1,-2)$ in the direction of a vector making a counter clockwise angle $\theta = \pi/2$ with the positive x-axis

$\longrightarrow$ $f_x = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2} \qquad f_x(-1,-2) = -4/9.$

$$f_y = \frac{(x+y)\times 1 - (x-4)}{(x+y)^2} = \frac{-2x}{(x+y)^2} \qquad f_y(-1,-2) = 2/9.$$

Unit vector u that makes an angle $\theta = \pi/2$ with the +ve x-axis is $u = \cos(\pi/2)\,i^\circ + \sin\pi/2\, j^\circ = j^\circ$.

$$D_u = -4/9 \times 0 + 2/9 \times 1 = \underline{2/9}$$

Let f be a function of 2 variables or 3 variables and let P denote the pt $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$. Assume f is differentiable at P.

1 If $\nabla f = 0$ then all directional derivatives of f at P are zero

2. If $\nabla f \neq 0$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. Value of this largest directional derivative is

$\underline{\|\nabla f\| \text{ at } P}$

3) If $\nabla f \neq 0$ at P, then among all possible directional derivatives of f at P., the derivative in the direction opposite to that of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P

1

Pbms

1 Find the divergence and curl of the vector field $F(x,y,z) = x^2y\,i + 2y^3z\,j + 3z\,k$.

$\longrightarrow$ $\operatorname{div} F = \nabla\cdot F = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)$

$= 2xy + 6y^2z + 3.$

$$\operatorname{curl} F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right) - j\left[\frac{\partial}{\partial x}(3z) - \frac{\partial}{\partial z}(x^2y)\right] + k\left(\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right)$$

$$-2y^3 i - x^2 k$$

---

2. S.T the divergence of the inverse square field $F(x,y,z) = \frac{c}{(x^2+y^2+z^2)^{3/2}}[xi + yj + zk]$ is zero.

$\longrightarrow$ $F = \frac{cx}{r^3}i + \frac{cy}{r^3}j + \frac{cz}{r^3}k$

$\left\{ \because r = xi + yj + zk,\ r^2 = x^2+y^2+z^2,\ (x^2+y^2+z^2)^{3/2} = (r^2)^{3/2} = r^3 \right\}$

$\operatorname{div} F = \nabla\cdot F = \frac{\partial}{\partial x}\left(\frac{cx}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{cy}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{cz}{r^3}\right)$

$$\boxed{\frac{\partial r}{\partial x} = x/r \quad \frac{\partial r}{\partial y} = y/r \quad \frac{\partial r}{\partial z} = z/r}$$

$\frac{\partial}{\partial x}\left(\frac{cx}{r^3}\right) = c\left[\frac{r^3\cdot 1 - x\times 3r^2\times\frac{\partial r}{\partial x}}{(r^3)^2}\right]$

$= c\left[\frac{r^3 - x\times 3r^2\times x/r}{r^6}\right] = c\left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$

|||ly $\frac{\partial}{\partial y}\left(\frac{cy}{r^3}\right) = c\left[\frac{1}{r^3} - \frac{3y^2}{r^5}\right]$ & $\frac{\partial}{\partial z}\left[\frac{cz}{r^3}\right] = c\left[\frac{1}{r^3} - \frac{3z^2}{r^5}\right]$

$\therefore \operatorname{div} F = c\left[\frac{1}{r^3} - \frac{x^2}{r^5} + 1/r^3 - \frac{y^2}{r^5} + 1/r^3 - \frac{z^2}{r^5}\right] = c\left[\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right]$

$= c\left[\frac{3}{r^3} - \frac{3r^2}{r^5}\right] = c\left(\frac{3}{r^3} - 3/r^3\right) = 0$

3) Find div F & curl F $F(x,y,z) = e^{xy} i - 2\cos y\, j + \sin^2 z\, k$

$\rightarrow$ $\text{div } F = \nabla \cdot F = \frac{\partial}{\partial x}(e^{xy}) + \frac{\partial}{\partial y}(-2\cos y) + \frac{\partial}{\partial z}(\sin^2 z)$

$= y e^{xy} + 2\sin y + 2\sin z \cos z$

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & -2\cos y & \sin^2 z \end{vmatrix} = i(0) - j(0) + k(0 - x e^{xy})$$

$= -x e^{xy} k$

---

4) $\bar{r} = xi + yj + zk$ S.T $\nabla\left(\frac{1}{\|r\|}\right) = -\frac{\bar{r}}{\|r\|^3}$.

$\rightarrow$ $r^2 = \|r\|^2 = x^2 + y^2 + z^2$ Let $r = \|r\|$.

We know that $\frac{\partial r}{\partial x} = x/r$ $\frac{\partial r}{\partial y} = y/r$ $\frac{\partial r}{\partial z} = z/r$

$\nabla(1/r) = i \frac{\partial}{\partial x}(1/r) + j \frac{\partial}{\partial y}(1/r) + k \frac{\partial}{\partial z}(1/r)$

$= i \times -1/r^2 \times \frac{\partial r}{\partial x} + j \times -1/r^2 \frac{\partial r}{\partial y} + k \times -1/r^2 \frac{\partial r}{\partial z}$

$= -\frac{i}{r^2} \times x/r - j/r^2 \times y/r - \frac{k}{r^2} \times z/r$

$= -\frac{1}{r^3}[xi + yj + zk] = \frac{-\bar{r}}{r^3}$

---

5) Use chain rule S.T $\nabla f(r) = \frac{f'(r)}{r}(\bar{r})$ when $\bar{r} = xi + yj + zk$ $r = \|r\|$

$\rightarrow$ $\nabla f(r) = i \frac{\partial}{\partial x} f(r) + j \frac{\partial}{\partial y} f(r) + k \frac{\partial}{\partial z} f(r)$

$= i \cdot f'(r) \cdot \frac{\partial r}{\partial x} + j\, f'(r) \frac{\partial r}{\partial y} + k\, f'(r) \frac{\partial r}{\partial z}$

$= i \cdot f'(r) \times x/r + j\, f'(r)\, y/r + k\, f'(r) \cdot z/r$

$= \frac{f'(r)}{r}[xi + yj + zk]$

$= \frac{f'(r)}{r} \bar{r}$

Conservative fields and Potential functions

A vector field F in 2-space or 3-space is said to be conservative in a region if it is the gradient field for some function ϕ in the region, i.e. $F = \nabla\phi$, the function ϕ is called potential function for F in the region.

Proms

1) The function $\phi(x,y,z) = xy + yz + xz$ is potential for the vector field F. Find the vector field F.

→ ϕ potential $\Rightarrow F = \nabla\phi$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

$$i(y+z) + j(x+z) + k(y+x)$$

2) Confirm ϕ is a potential function for F where $\phi(x,y) = 2y^2 + 3x^2y - xy^4$ & $F(x,y) = (6xy - y^4)i + (4y + 3x^2 - 4xy^3)j$

→ $$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(6xy - y^4) + j[4y + 3x^2 - 4xy^3] = F.$$

3) Determine a so that $(x+3y)i + (y-2z)j + (x+az)k$ is solenoidal.

→ F is solenoidal $\Rightarrow \operatorname{div} F = 0$, $\nabla\cdot F = 0$

$$\nabla\cdot F = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$= 1 + 1 + a = 0 \quad 2 + a = 0 \quad a = -2$

Conservative Vector field $\nabla \times F = 0$.

or $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ //.

Line integrals

If C is a smooth curve parametrized by

$r(t) = x(t)i + y(t)j \quad (a \le t \le b)$.

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\, \|r'(t)\|\, dt$$

IIIly $\quad r(t) = x(t)i + y(t)j + z(t)k \quad (a \le t \le b)$

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\, \|r'(t)\|\, dt$$

Pbms

1) Using given parametrization evaluate the line integral

$\int_C (1+xy^2)\,ds$

(a) $C: r(t) = ti + 2tj \quad 0 \le t \le 1$

(b) $C: r(t) = (1-t)i + (2-2t)j \quad 0 \le t \le 1$

→ (a) Given $r(t) = ti + 2tj \qquad x(t) = t, \quad y(t) = 2t$.

$r'(t) = i + 2j \qquad \|r'(t)\| = \sqrt{5}$

$$\int_C (1+xy^2)\,ds = \int_0^1 \left[1 + t(2t)^2\right] \sqrt{5}\, dt = \sqrt{5}\int_0^1 1 + 4t^3\, dt$$

$$\sqrt{5}\left(t + \frac{4t^4}{4}\right)_0^1 = \underline{\underline{2\sqrt{5}}}.$$

(b) $r(t) = (1-t)i + (2-2t)j \qquad x(t) = 1-t, \quad y(t) = 2(1-t)$

$r'(t) = -i - 2j \qquad \|r'(t)\| = \underline{\underline{\sqrt{5}}}$

$$\int_C (1+xy^2)\,ds = \int_0^1 \left(1 + (1-t)(2(1-t))^2\right) \sqrt{5}\, dt$$

$$= \sqrt{5}\int_0^1 1 + 4(1-t)^3\, dt$$

$$= \sqrt{5}\int_0^1 1 + 4\left[1 - 3t + 3t^2 - t^3\right] dt$$

$$= \sqrt{5}\left[t + 4t - \frac{12t^2}{2} + 12\frac{t^3}{3} - \frac{4t^4}{4}\right]_0^1 = \underline{\underline{2\sqrt{5}}}$$

2) ~~Consider~~ Evaluate the line integral $\int 3xyz \, ds$ where the curve C has parametrization $x=t$, $y=t^2$, $z=\frac{2}{3}t^3$ $0\le t\le 1$

$\longrightarrow$ $r(t) = t\,i + t^2 j + \frac{2}{3}t^3$ $\quad r'(t) = i + 2t\,j + 2t^2 k$

$\|r'(t)\| = \sqrt{1+4t^2+4t^4} = \sqrt{(1+2t^2)^2} = \underline{1+2t^2}$.

$$\int 3xyz\,ds = \int_0^1 3 \cdot t \times t^2 \times \frac{2}{3}t^3 \times (1+2t^2)\,dt$$

$$= \int_0^1 2t^6(1+2t^2)\,dt = \int_0^1 2t^6 + 4t^8\,dt$$

$$\left[2\frac{t^7}{7} + 4\frac{t^9}{9}\right]_0^1 = \frac{2}{7} + 4/9 = \underline{\frac{46}{63}}.$$

3) Evaluate the line integral $\int (xy+z^3)\,ds$ from $(1,0,0)$ to $(-1,0,\pi)$ along the helix 'c' that is represented by the parametric equations $x=\cos t$ $y=\sin t$ $z=t$ $(0\le t\le \pi)$

$\longrightarrow$ $r(t) = \cos t\, i + \sin t\, j + t\,k$ $\quad \|r'(t)\| = \sqrt{(-\sin t)^2+(\cos t)^2+1} = \underline{\sqrt{2}}$

$r'(t) = -\sin t\, i + \cos t\, j + k$

$$\int f(x,y,z)\,ds = \int_0^{\pi} \left[\cos t \sin t + t^3\right] \sqrt{2}\,dt$$

$$= \int_0^{\pi} \left[\frac{\sin 2t}{2} + t^3\right]\sqrt{2}\,dt$$

$$= \sqrt{2}\left[-\frac{\cos 2t}{4} + \frac{t^4}{4}\right]_0^{\pi}$$

$$= \sqrt{2}\left[-\frac{1}{4} + \frac{\pi^4}{4} - \left(-\frac{1}{4}\right)\right] = \underline{\frac{\sqrt{2}\,\pi^4}{4}}.$$

Line integral of Vector Valued functions

Let $x = x(t)$, $y = y(t)$, $z = z(t)$ $\quad a\le t\le b$ be the parametric eqns for C in which the orientation of C is in the direction of increasing t,

Pbms

1 Evaluate $\int_C 3xy\,dy$, where C is a line segment joining $(0,0)$ and $(1,2)$ with the given orientation.

a) [Oriented from $(0,0)$ to $(1,2)$] $C: t\hat{i} + 2t\hat{j}$

b) [Oriented from $(1,2)$ to $(0,0)$] $C: (1-t)\hat{i} + (2-2t)\hat{j}$

→ a) $C: t\hat{i} + 2t\hat{j}$ $\quad x = t \quad y = 2t$

$dy = 2dt$

$$\int_0^1 3 \times t \times 2t \times 2dt = 12\int_0^1 t^2 dt = 12\left[\frac{t^3}{3}\right]_0^1 = \underline{\underline{4}}$$

b) $x = 1-t \quad y = 2-2t \quad dy = -2dt$

$$\int_1^0 3(1-t)(2-2t) \times -2\,dt = -12\int_1^0 (1-t)^2 dt = \underline{+4}.$$

2) Evaluate $\int_C 2xy\,dx + (x^2+y^2)\,dy$ along the circular arc C given by $x = \cos t$, $y = \sin t$ $(0 \le t \le \pi/2)$

→ $x = \cos t \quad dx = -\sin t\,dt$

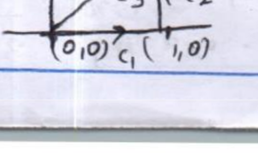
$y = \sin t \quad dy = \cos t\,dt$

$$\int_0^{\pi/2} 2\cos t \sin t \times -\sin t\,dt + [\cos^2 t + \sin^2 t] \times \cos t\,dt$$

$$\int_0^{\pi/2} -2\sin^2 t \cos t\,dt + \cos t\,dt$$

$$= -\frac{2\sin^3 t}{3}\Big]_0^{\pi/2} + \sin t\Big]_0^{\pi/2} = -3/2 + 1 = \underline{+1/3}.$$

3). Evaluate $\int x^2 y\,dx + x\,dy$ where C is a triangular path shown in figure.



$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

Along C_1: $y=0$, $dy=0$, $x \to 0$ to 1

$$\int_{C_1} = \int_0^1 x^2 \times 0\, dx + x \times 0 = \underline{\underline{0}}$$

Along C_2: $(1,0)$ to $(1,2)$

$$\frac{y-0}{2-0} = \frac{x-1}{0}$$

$\Rightarrow x-1=0$, $x=1$, $dx=0$

$y \to 0$ to 2

$$\int_{C_2} = \int_0^2 x^2 y \times 0 + x\, dy = \int_0^2 dy = [y]_0^2 = \underline{\underline{2}}$$

Along C_3: $(1,2)$ to $(0,0)$

$$\frac{y-2}{0-2} = \frac{x-1}{0-1} \Rightarrow y-2 = 2(x-1)$$

$y = 2x$, $dy = 2dx$, $x \to 1$ to 0

$$\int_{C_3} = \int_1^0 x^2 \times 2x\, dx + x \times 2dx$$

$$\int_1^0 2x^3\, dx + 2x\, dx = \left[\frac{2x^4}{4} + \frac{2x^2}{2}\right]_1^0$$

$$= \underline{\underline{-3/2}}$$

$$\int_C f = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + 2 - 3/2 = \underline{\underline{\frac{1}{2}}}.$$

Work Integral [Work done]

$$\text{Work done} = \int_C F \cdot dr$$

, F is a continuous vector field & C is smooth oriented curve.

Line integral $\rightarrow \int_C f \cdot dr$

Probs

1) Find the work done in moving a particle in the force field $F = 3x^2 i + (2xz - y) j + z k$ along the curve oriented by $x^2 = 4y$, $3x^3 = 8z$ from $x=0$ to $x=2$.

$\rightarrow$ Workdone $= \int f \cdot dr$

$r = x i + y j + z k$

$dr = dx i + dy j + dz k$

$$\int f \cdot dr = \int \left[3x^2 i + (2xz - y) j + 3k\right] \cdot \left[dx i + dy j + dz k\right]$$

$$= \int 3x^2 dx + (2xz - y) dy + z dz.$$

Here $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$, $dy = \frac{x}{2} dx$.

$3x^3 = 8z \Rightarrow z = \frac{3}{8} x^3$ $dz = 9/8 \, x^2 dx$

$$\Rightarrow \int f \cdot dr = \int_0^2 3x^2 dx + \left(2x \times \frac{3}{8} x^3 - \frac{x^2}{4}\right) \frac{x}{2} dx + \frac{3x^3}{8} \, 9/8 \, x^2 dx.$$

$$= \int_0^2 3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27 x^5}{64} dx$$

$$\left[\frac{3x^3}{3} + \frac{3 \cdot x^6}{8 \cdot 6} - \frac{x^4}{8 \cdot 4} + \frac{27 x^6}{64 \times 6}\right]_0^2$$

$$= 8 + \frac{64}{16} - \frac{16}{32} + \frac{27 \times 64}{64 \times 6} = \underline{16}$$

Independence of Path

$F(x,y) = f(x,y) i + g(x,y) j$ is a conservative vector field in some open region D containing the pts (x_0, y_0) and (x_1, y_1) and that f & g are continuous in this region. If $F(x,y) = \nabla\phi(x,y)$ and if C starts from (x_0, y_0) and ends at (x_1, y_1) that lies in D then

$\int_C F(x,y) \cdot dr$ or $\int_C \nabla\phi \cdot dr = \phi(x_1, y_1) - \phi(x_0, y_0)$

E doesnot depends on path.

Pbms

1) Confirm that $F(x,y) = yi + xj$ is conservative then find $\int_{(0,0)}^{(1,1)} F \cdot dr$?

$\Longrightarrow$ $F(x,y) = yi + xj$

Here $f = y$ $\quad g = x$

$\frac{\partial f}{\partial y} = 1 \quad \frac{\partial g}{\partial x} = 1$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \Longrightarrow F$ is conservative

$\therefore F = \nabla \phi \quad yi + xj = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y}$

$\frac{\partial \phi}{\partial x} = y \Rightarrow \phi = xy + c_1$

$\frac{\partial \phi}{\partial y} = x \Rightarrow \phi = xy + c_2$

$\therefore \underline{\phi = xy}$

$\therefore \int_{(0,0)}^{(1,1)} F \cdot dr = \phi(1,1) - \phi(0,0) = 1 - 0 = \underline{1}$

2) S.T $F = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative vector field. Find its scalar potential. Find workdone in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$

$\rightarrow$ $f = 2xy + z^3 \quad g = x^2 \quad h = 3xz^2$

$$\text{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = i\left[\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right] - j\left[\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right] + k\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$= i(0 - 0) - j[3z^2 - 3z^2] + k[2x - 2x]$

F is conservative vector field $= \underline{0}$

$F = \nabla \phi \Rightarrow (2xy + z^3)i + x^2 j + 3xz^2 k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

$\frac{\partial \phi}{\partial x} = 2xy + z^3 \xRightarrow{\text{integ}} \phi = x^2 y + xz^3 + c_1$

$\frac{\partial \phi}{\partial y} = x^2 \Rightarrow \phi = x^2 y + c_2$

$\frac{\partial \phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + c$

$\phi = x^2 y + xz^3 + c$

Potential fun

Module IV
Multivariable Calculus - Integration

Double integrals

A double integral can be evaluated by two successive integrations. We evaluate it w.r.to one variable (treating the other variable as constant) and reduce it to an integral of one variable.

i.e. $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$
$$= \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

[Rectangular region]

Problems

1 $$\int_1^3 \int_2^4 40-2xy\,dy\,dx$$
$$= \int_1^3 \left[\int_2^4 40-2xy\,dy \right] dx$$
$$= \int_1^3 \left[40y - \frac{2xy^2}{2} \right]_2^4 dx = \int_1^3 80-12x\,dx$$
$$= \left. 80x - \frac{12x^2}{2} \right]_1^3 = 112$$

---

2 Evaluate the double integral $\iint_R y^2x\,dA$ over the rectangle $R = \{(x,y) \mid -3 \le x \le 2,\ 0 \le y \le 1\}$

$\Rightarrow$ $$= \iint_R y^2x\,dA = \int_{-3}^2 \int_0^1 y^2x\,dy\,dx. = -5/6$$

---

3 $\int_1^a \int_1^b \frac{1}{xy}\, dy\, dx = \int_1^a \frac{1}{x} dx \int_1^b \frac{1}{y} dy$

$= (\log x)_1^a (\log x)_1^b = \log a \log b$

---

4 $\int_{\pi/2}^{\pi} \int_1^2 x \sin(xy)\, dy\, dx$

$\Rightarrow \int_{\pi/2}^{\pi} \left[\cancel{x} - \frac{\cos(xy)}{\cancel{x}}\right]_1^2 dx = \int_{\pi/2}^{\pi} -\cos 2x + \cos x\, dx$

$= -\frac{\sin 2x}{2} + \sin x \Big]_{\pi/2}^{\pi} = -1$

---

5 Evaluate. $\int_0^1 \int_{-x}^{x^2} y^2 x\, dy\, dx.$

$\rightarrow \int_0^1 x \frac{y^3}{3}\Big]_{-x}^{x^2} dx = \int_0^1 \frac{x^7}{3} + \frac{x^4}{3} dx = \frac{13}{120}.$

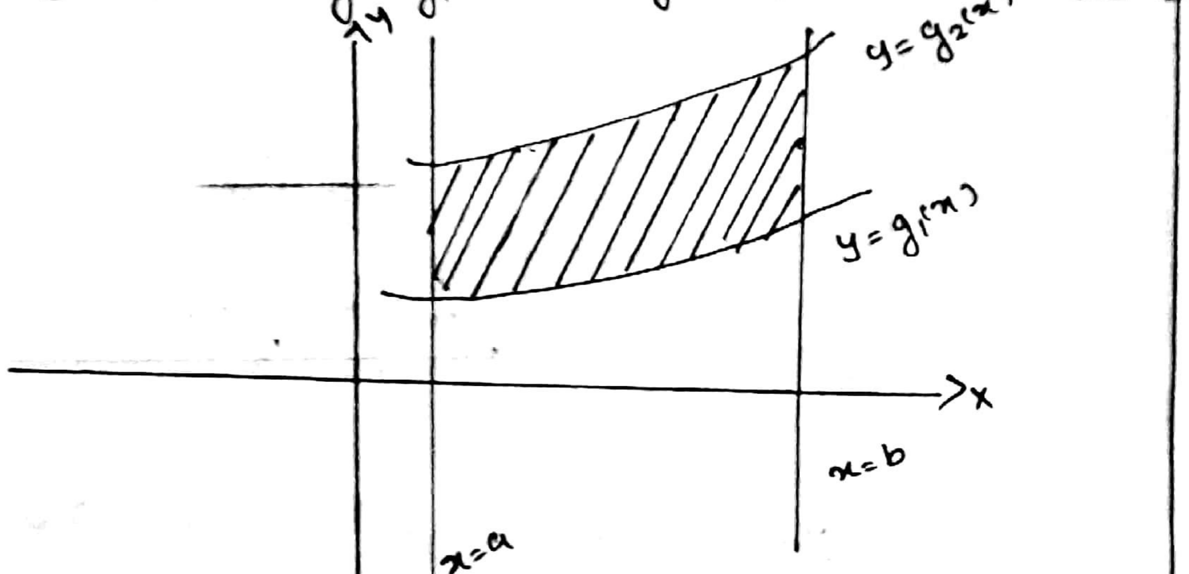
Fubini's theorem

Let R be a rectangle defined by the inequalities $a \le x \le b$, $c \le y \le d$. If $f(x,y)$ is continuous on this rectangle then,

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Double integral over non rectangular region

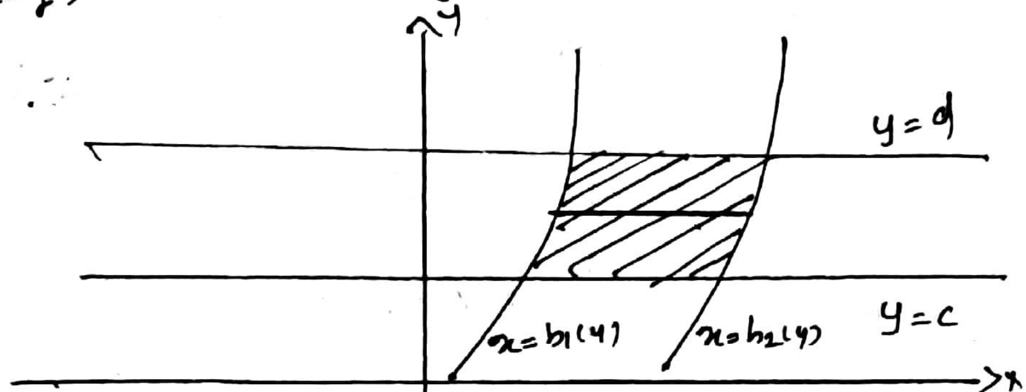
Type 1 Region : It is a region bounded on the left and right by the vertical line $x=a$ and $x=b$ and is bounded below and above by the curves $y=g_1(x)$ $y=g_2(x)$, $g_1(x) \le g_2(x)$ $a \le x \le b$



Since x is fixed we draw vertical line in the region R at an arbitrary fixed value. The line crosses the boundary of R twice. The lower point of the intersection is on the curve $y=g_1(x)$ higer point is on the curve $y=g_2(x)$. These two intersection determines lower and upper limit of y. Imagine move the line to left and then to right. left most position where the line intersect the region R is $x=a$ and the right position is $x=b$. This determines the limit of x.

Type 2 Region

It is a region bounded below and above by the horizontal lines $y=c$ and $y=d$ and bounded on left and right by the continuous curves $x=h_1(y)$ and $x=h_2(y)$ $\therefore$ $h_1(y) \le h_2(y)$ for $c \le y \le d$.



Since y is fixed we draw a horizontal line in the region R. The line also crosses the boundary twice. The left side is on the curve $x=h_1(y)$ and right side is on the curve $x=h_2(y)$. Move the line from bottom to top. from $y=c$ to $y=d$.

---

y — Constant , x = Variable

Problems

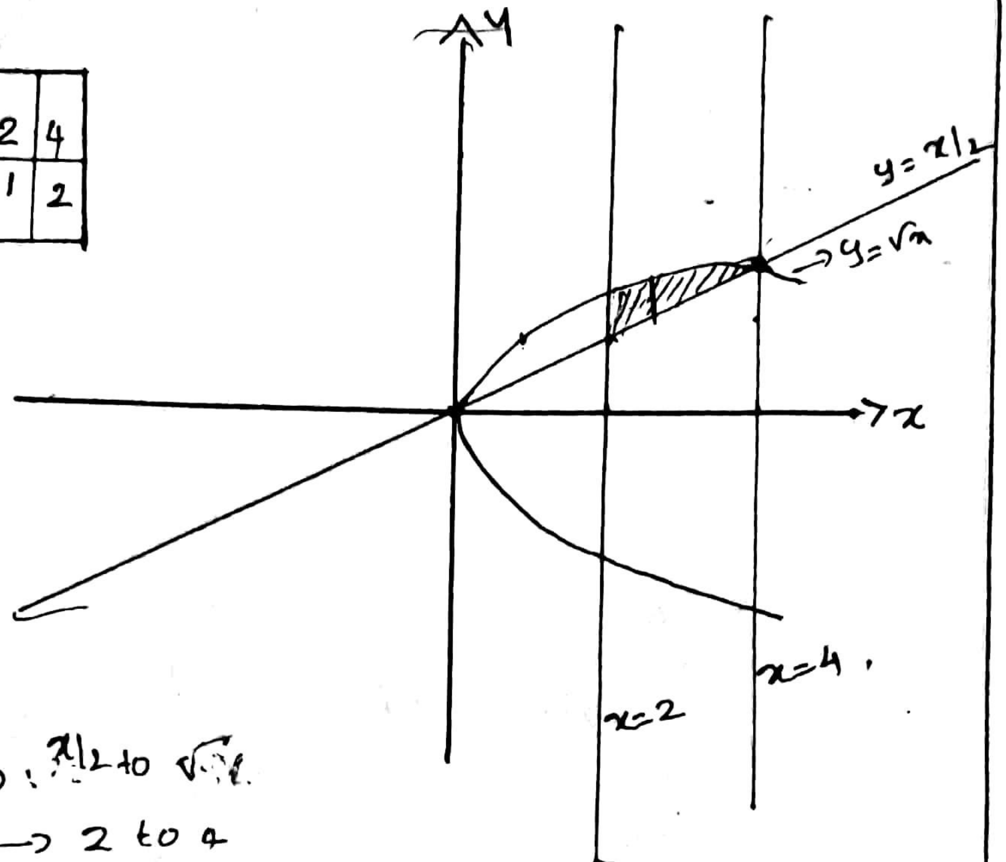
1 Evaluate $\iint_R xy\, dA$, R is enclosed b/w $y = x/2$, $y = \sqrt{x}$, $x = 2$, $x = 4$

$\Rightarrow$ $y = x/2$

x	0	2	4
y	0	1	2

$y = \sqrt{x}$

x	0	1	4
y	0	1	2



Type I

$y \to x/2$ to $\sqrt{x}$

$x \to 2$ to 4

$$\int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 x \frac{y^2}{2} \Big]_{x/2}^{\sqrt{x}} dx.$$

$$= \int_2^4 \frac{x^2}{2} - \frac{x^3}{8}\, dx = \frac{11}{6}.$$

---

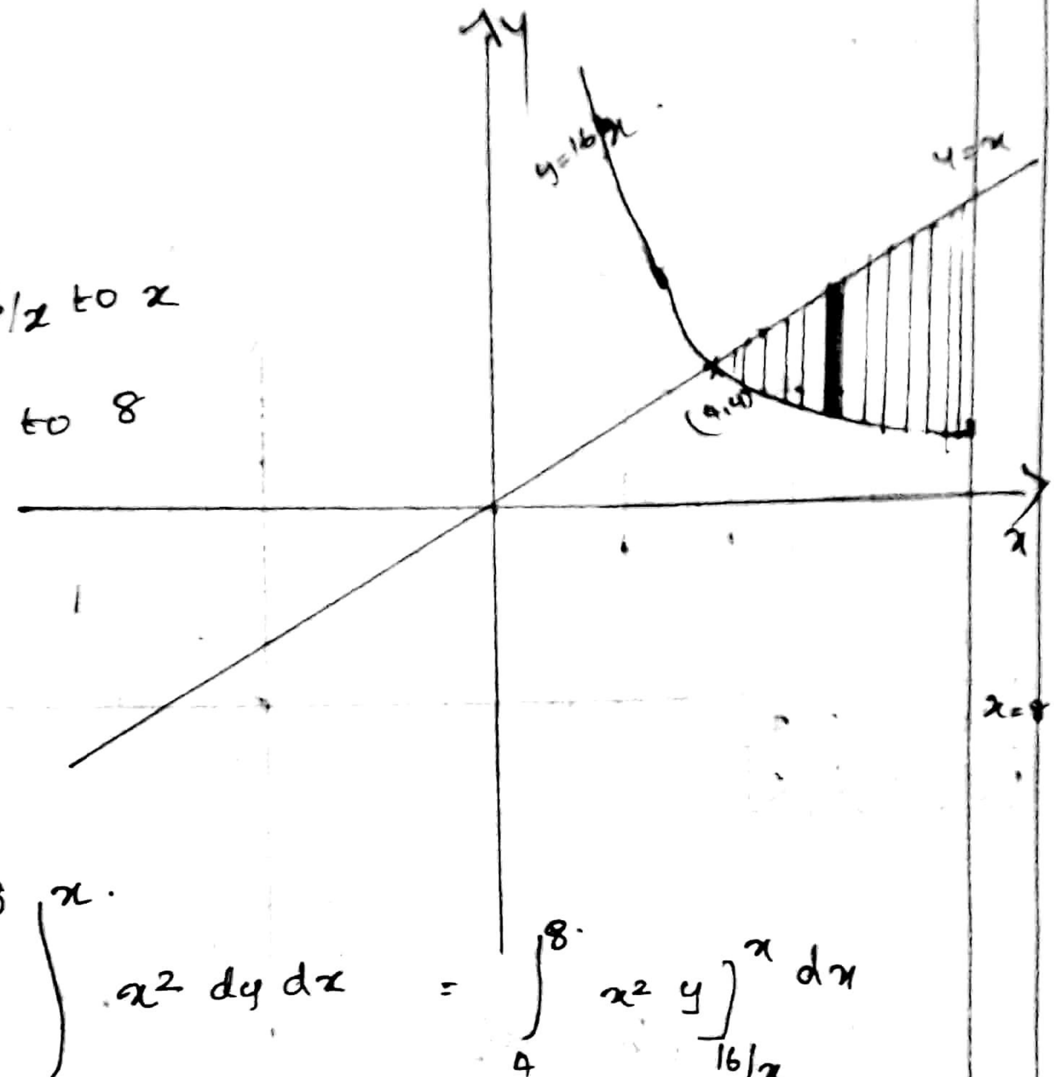
2 Evaluate $\iint_R x^2\, dA$. bounded by $y = 16/x$, $y = x$, $x = 8$.

$\Rightarrow$ $y = \frac{16}{x}$

x	2	4	8
y	8	4	2

$y = x$

x	0	4	8
y	0	4	8



$y \to 16/x$ to x

$x \to 4$ to 8

$$\int_4^8 \int_{16/x}^{x} x^2 \, dy \, dx = \int_4^8 x^2 y \Big]_{16/x}^{x} dx$$

$$= \int_4^8 x^3 - 16x \, dx = 576.$$

3 Evaluate $\iint_R 2x - y^2$ over the rectangular region is enclosed b/w $y = -x+1$, $y = x+1$, $y = 3$

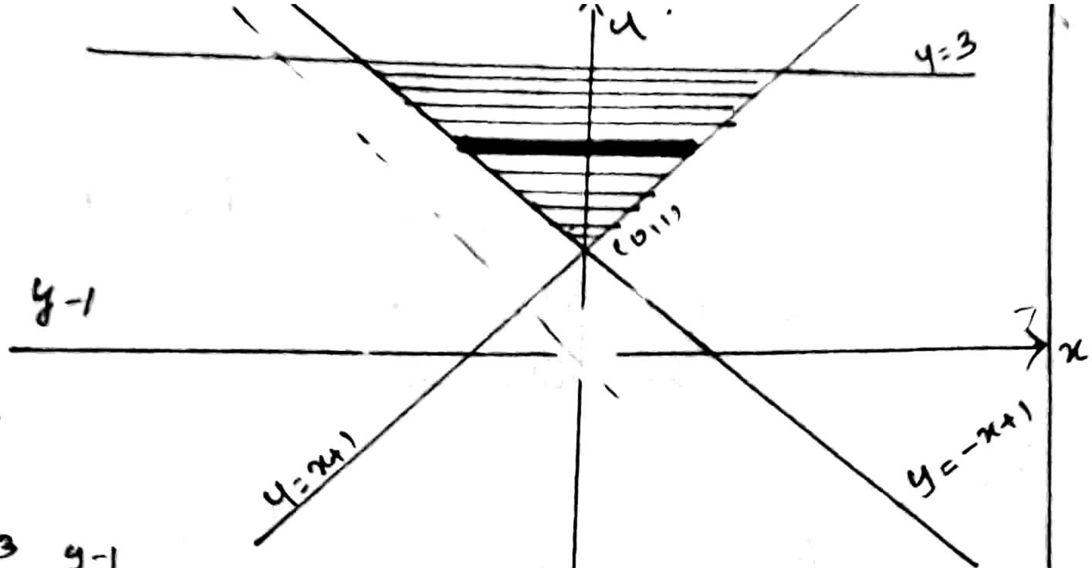
$\Rightarrow$ $y = -x+1$

x	0	1	2
y	+1	0	-

$y = x+1.$

x	0	1	2
y	1	2	3

4



Type 2

$x \rightarrow 1-y$ to $y-1$

$y \rightarrow 1$ to 3

$$\int_1^3 \int_{1-y}^{y-1} (2x-y^2)\,dx\,dy = \int_1^3 \left[x^2 - y^2 x\right]_{1-y}^{y-1} dy$$

$$= \int_1^3 2y^2 - 2y^3\,dy = \left[\frac{2y^3}{3} - \frac{2y^4}{4}\right]_1^3 = -\frac{68}{3}$$

Use a double integral find the area of Region R enclosed b/w a parabola $y = \frac{x^2}{2}$ and the line $y = 2x$

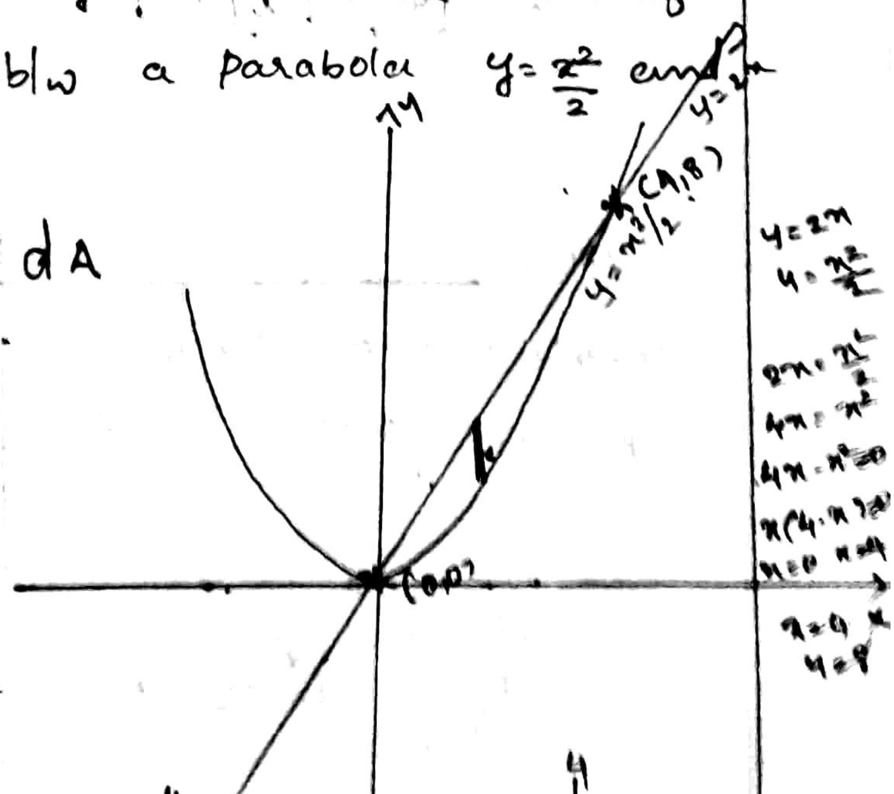
→ $\text{AREA} = \iint_R dA$

$y = \frac{x^2}{2}$

x	0	2	4
y	0	2	8

$y = 2x$

x	0	1	2
y	0	2	4



$y = 2x$
$y = \frac{x^2}{2}$
$2x = \frac{x^2}{2}$
$4x = x^2$
$4x - x^2 = 0$
$x(4-x) = 0$
$x=0 \quad x=4$
$x=4$
$y=8$

$y \rightarrow \frac{x^2}{2}$ to $2x$

$x \rightarrow 0$ to 4.

$$\text{Area} = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 \left(y\right)_{x^2/2}^{2x} dx = \int_0^4 2x - \frac{x^2}{2}\,dx$$

$= 16$

Reversing the order of integration

Sometimes the evaluation of a double integral can be simplified by reversing the order of integration.

Problems

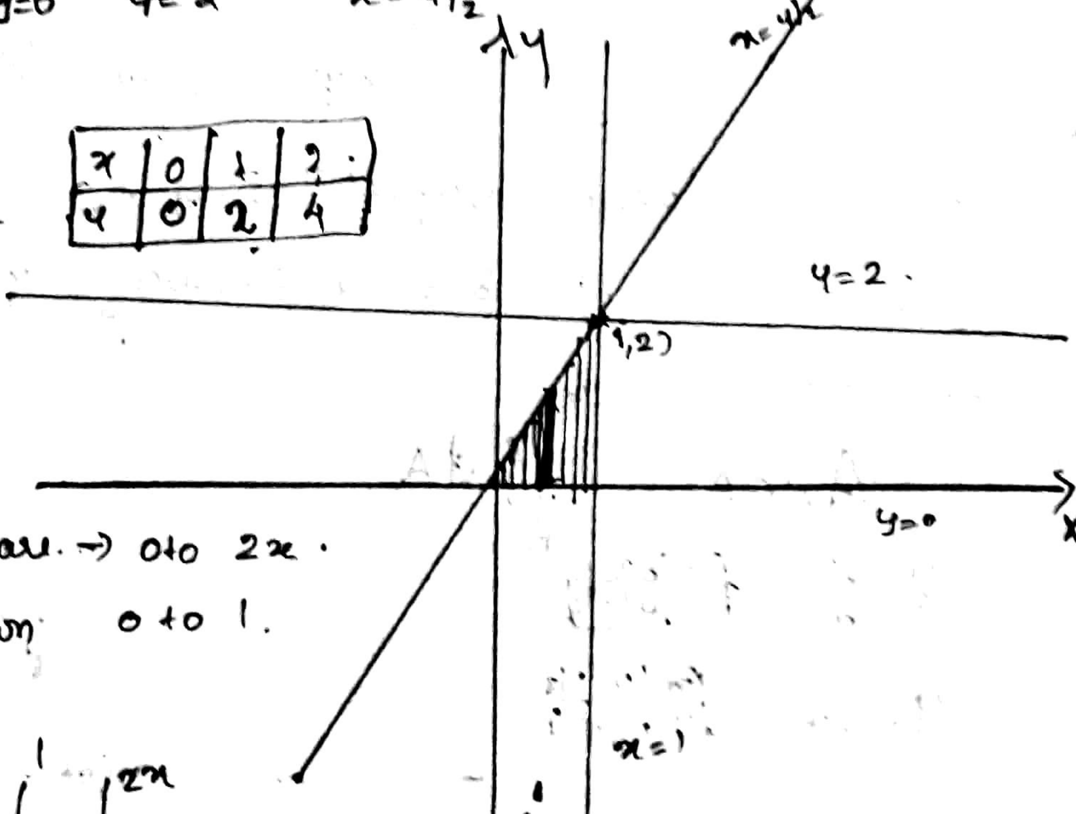
1 Evaluate integral $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$

Type 2 region changes to Type 1 region

$y=0 \quad y=2 \quad x = y/2 \quad x=1.$

$x = y/2$

x	0	1	2
y	0	2	4



Type 1

$y \to$ var. $\to$ 0 to $2x$.

$x \to$ con 0 to 1.

$$\int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 e^{x^2} (y)_0^{2x}\, dx$$

$$= \int_0^1 e^{x^2}\, 2x\, dx = \left(e^{x^2}\right)_0^1 = \underline{e-1}$$

2. Sketch the region of integration and evaluate the integral $\int_1^2 \int_y^{y^2} dx\,dy$ by changing the Order of integration. 5

Type 2 Region. Changes to, Type 1 Region

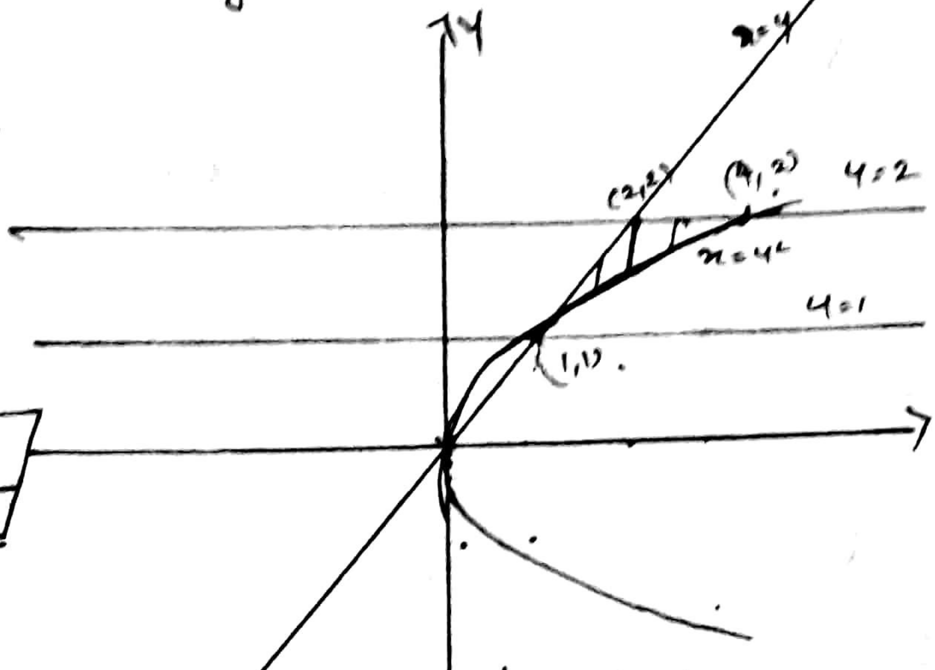
$y=1$ $y=2$ $x=y$ $x=y^2$

$x=y$

x	0	1
y	0	1

$x=y^2$

x	0	1	1
y	0	1	-1



Split the region to two part

1^{st} part. $y \to \sqrt{x}$ to x

$x \to 1$ to 2.

2^{nd} part $y \to \sqrt{x}$ to 2

$x \to 2$ to 4.

$$\iint_R dx\,dy = \int_1^2 \int_{\sqrt{x}}^{x} dy\,dx + \int_2^4 \int_{\sqrt{x}}^{2} dy\,dx.$$

$$= \int_1^2 (y)_{\sqrt{x}}^{x}\,dx + \int_2^4 (y)_{\sqrt{x}}^{2}\,dx.$$

$$= \int_1^2 x-\sqrt{x} + \int_2^4 2-\sqrt{x}\,dx = \left[\frac{x^2}{2}-\frac{x^{3/2}}{3/2}\right]_1^2 + \left[2x-\frac{x^{3/2}}{3/2}\right]_2^4$$

Volume $= \iint f(x,y)\, dA$ Where $z = f(x,y)$

1) Find the volume of solid bounded by the cylinder $x^2+y^2=4$ $y+z=4$ $z=0$.

$\Rightarrow$ $y+z=4$ $z=4-y$

Volume $= \iint f(x,y)\, dA$

$= \iint 4-y\, dA$.

$x^2+y^2=4$

$y \to -\sqrt{4-x^2}$ to $\sqrt{4-x^2}$ $x \to -2$ to 2

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4-y\, dy\, dx.$$

$$= \int_{-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= \int_{-2}^{2} 4\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right) - \left[-4\sqrt{4-x^2} - \left(\frac{4-x^2}{2}\right)\right] dx$$

$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx = 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}(x/2)\right]_{-2}^{2}$$

$$= 8\left[\sqrt{4-4} + 2\sin^{-1}(1) - \left(-\sqrt{4-4} + 2\sin^{-1}(-1)\right)\right]$$

$$= 8\left[2\sin^{-1}(1) + 2\times\sin^{-1}(1)\right] = 32\sin^{-1}(1)$$

$$= 32 \times \pi/2$$

$$= 16\pi$$

6.

Triple Integrals

A Single integral of a function f(x) is defined over a finite closed interval on the x-axis, and a double integral of a function f(x,y) is defined over a finite closed region R in the xy-plane.

A triple integral of f(x,y,z) over a closed solid region G in an xyz-co-ordinate system.

$$\boxed{\text{Volume} = \iiint dV}$$

Pbms

1 Evaluate $\iiint_G 12xy^2z^3\,dV$ over the rectangular blocks G defined by the inequalities

$-1 \le x \le 2,\ 0 \le y \le 3,\ 0 \le z \le 2.$

$$\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12\,xy^2z^3\,dz\,dy\,dx$$

$$= \int_{-1}^{2}\int_{0}^{3} 12xy^2 \left.\frac{z^4}{4}\right]_0^2 dy\,dx$$

$$\int_{-1}^{2}\int_{0}^{3} 48\,xy^2\,dy\,dx = \int_{-1}^{2} 48\,x \left.\frac{y^3}{3}\right]_0^3 dx$$

$$= 432\int_{-1}^{2} x\,dx$$

$$= 432\left[\frac{x^2}{2}\right]_{-1}^{2} = \underline{648}$$

2 $\displaystyle\int_0^1\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy$

$$= \int_0^1\int_{-1}^{y^2} [xyz]_{-1}^{z}\,dz\,dy$$

$$\int_0^1 \left. y\frac{z^3}{3} + y\frac{z^2}{2}\right]_{-1}^{y^2} dy$$

$$= \int_0^1 \frac{y^7}{3} + \frac{y^5}{2} + y/3 - y/2 \, dy$$

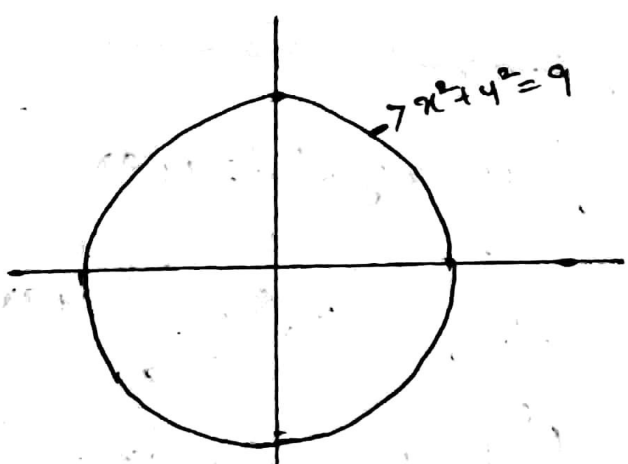
$$= \left. \frac{y^8}{8\times3} + \frac{y^6}{6\times2} + \frac{y^2}{2\times3} - \frac{y^2}{2\times3}\right]_0^1 = 1/24.$$

---

3. Use a triple integral to find the volume of the solid within the cylinder $x^2+y^2=9$ and between the planes $z=1$ and $x+z=5$.

$\rightarrow$ Volume $= \iiint dv$

Here $z \rightarrow 1$ to $5-x = 5-r\cos\theta$.



Sub $x = r\cos\theta$ $\quad$ $dx\,dy = r\,dr\,d\theta$

$y = r\sin\theta$.

$r \rightarrow 0$ to 3

$\theta \rightarrow 0$ to 2π

$$\therefore V = \int_0^{2\pi}\int_0^3\int_1^{5-r\cos\theta} dz \cdot r\,dr\,d\theta.$$

$$\int_0^{2\pi}\int_0^{3}\left[z\right]_0^{5-r\cos\theta} \cdot r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^{3}\left[5-r\cos\theta-1\right] r.\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^{3}\left[4-r\cos\theta\right] r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^{3} 4r-r^2\cos\theta\, dr\, d\theta .$$

$$= \int_0^{2\pi}\left[\frac{4r^2}{2}-\frac{r^3}{3}\cos\theta\right]_0^3 d\theta$$

$$= \int_0^{2\pi} 18-9\cos\theta\, d\theta$$

$$= \left[18\theta-9\sin\theta\right]_0^{2\pi} = \underline{36\pi.}$$

4. Evaluate $\iiint_G xyz\, dv$ when G is the solid in the first octant that is bounded by the parabolic cylinder $z=3-x^2$ and the planes $z=0$ $y=x$ and $y=0$

→ <u>Limits</u>

$z \to 0$ to $3-x^2$.

$y \to 0$ to x

$x \to 0$ to $\sqrt{3}$

$\left[\begin{array}{l} y=0,\ y=x \Rightarrow x=0 \\ z=0\ \ z=3-x^2 \Rightarrow x^2=3 \\ x=\sqrt{3} \end{array} \right]$

$$\iiint_G xyz\, dv = \int_0^{\sqrt{3}}\int_0^{x}\int_0^{3-x^2} xyz\, dz\, dy\, dx$$

$$\int_0^{\sqrt{3}} \int_0^{x} xy \left.\frac{z^2}{2}\right]_0^{3-x^2} dy\,dx$$

$$\frac{1}{2}\int_0^{\sqrt{3}} \int_0^{x} xy\,(3-x^2)^2\,dy\,dx$$

$$\frac{1}{2}\int_0^{\sqrt{3}} \int_0^{x} xy\,(9-6x^2+x^4]\,dy\,dx.$$

$$\frac{1}{2}\int_0^{\sqrt{3}} \int_0^{x} 9xy - 6x^3y + x^5y\,dy\,dx$$

$$\int_0^{\sqrt{3}} \left[9x\frac{y^2}{2} - 6x^3\frac{y^2}{2} + x^5\frac{y^2}{2}\right]_0^x dx$$

$$\frac{1}{4}\int_0^{\sqrt{3}} 9x^3 - 6x^5 + x^7\,dx = \frac{27}{32}.$$

---

5 Use a triple integral to find the volume of the solid in the first Octant bounded by the Co-ordinate planes and the plane $3x+6y+4z=12$

$\rightarrow$ $V = \iiint dv.$

$$Z = \frac{12-3x-6y}{4}$$

$Z=0 \Rightarrow \frac{12-3x-6y}{4} = 0 \Rightarrow y = \frac{4-x}{2}.$

Limits

$z \rightarrow 0$ to $\frac{12-3x-6y}{4}$

$y \rightarrow 0$ to $\frac{4-x}{2}$

$x \rightarrow 0$ to 4

$$V = \int_0^4 \int_0^{\frac{4-x}{2}} \int_0^{\frac{12-3x-6y}{4}} dz\, dy\, dx$$

$$= \int_0^4 \int_0^{\frac{4-x}{2}} \frac{12-3x-6y}{4}\, dy\, dx.$$

$$\frac{1}{4} \int_0^4 \int_0^{\frac{4-x}{2}} 12-3x-6y \; dy\, dx$$

$$= \frac{1}{4} \int_0^4 12y - 3xy - \frac{6y^2}{2} \Big]_0^{\frac{4-x}{2}} dx$$

$$= \frac{1}{4} \int_0^4 12\frac{(4-x)}{2} - 3x\frac{(4-x)}{2} - \frac{6\left(\frac{4-x}{2}\right)^2}{2}.$$

$$= \frac{1}{4} \int_0^4 24 - 6x - 6x + \frac{3x^2}{2} - 3\left(4 - 2x + \frac{x^2}{4}\right) dx$$

$$\frac{1}{4} \int_0^4 12 - 6x + \frac{3x^2}{4}\, dx$$

$$\frac{1}{4} \left(12x - \frac{6x^2}{2} + \frac{3x^3}{3 \times 4}\right)_0^4 = \underline{\underline{4}}$$

Mass of Lamina

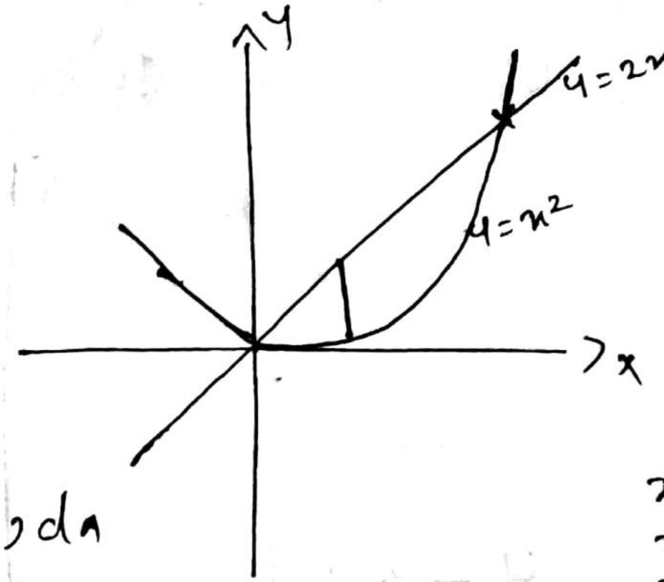
If $\rho(x,y)$ is a continuous density function of a lamina in the plane region R, then mass of lamina is

$$m = \iint_R \rho(x,y)\, dA$$

Pbms

1) find the mass of the region that is bounded by the line $y = 2x$ and the parabola $y = x^2$ if the density function is $\rho(x,y) = x$.

→



$y \to x^2$ to $2x$

$x \to 0$ to 2

$y = x^2$
$y = 2x$
$x^2 = 2x$
$x^2 - 2x = 0$
$x(x-2) = 0$
$x = 0,\ x = 2$

$$M = \int_0^2 \int_{x^2}^{2x} \rho(x,y)\, dA$$

$$= \int_0^2 \int_{x^2}^{2x} x\, dy\, dx \quad = \int_0^2 [xy]_{x^2}^{2x}\, dx$$

$$= \int_0^2 2x^2 - x^3\, dx = \left.\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{4}{3}$$

Centre of mass of Lamina

centre of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

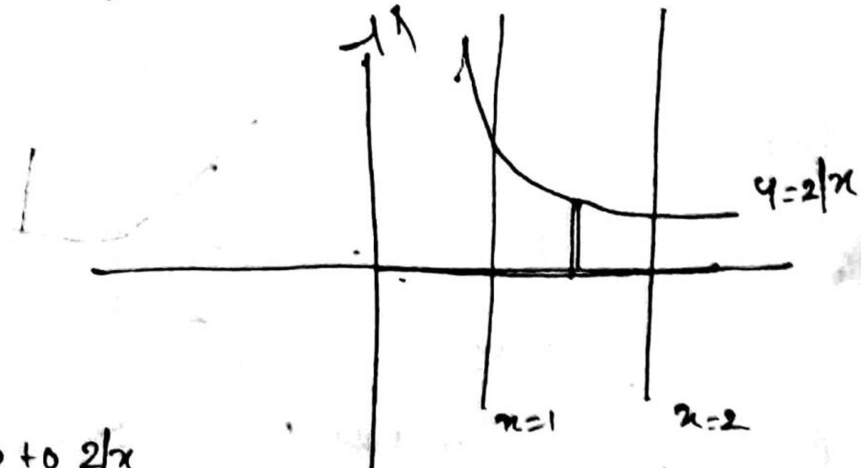
$$M_x = \iint_R y \, \rho(x,y) \, dA$$

$$M_y = \iint_R x \, \rho(x,y) \, dA.$$

Pbm

1) Find the mass and center of mass of the lamina bounded by $y = 2/x$, $y=0$ $x=1$, $x=2$ with density $\rho = kx^2$.

→



$y \to 0$ to $2/x$

$x \to 1$ to 2

$$\text{Mass } M = \int_1^2 \int_0^{2/x} \rho(x,y) \, dA$$

$$= \int_1^2 \int_0^{2/x} kx^2 \, dy \, dx$$

$$\int_1^2 \int_0^{2/x} kx^2 \, dy \, dx$$

$$\int_1^2 \left. k x^2 y \right]_0^{2/x} dx$$

$$\int_1^2 2k\, x \, dx = 2k \left. \frac{x^2}{2} \right]_1^2$$

$$= \underline{\underline{3k}}.$$

$$M_x = \iint_R y\, \rho(x,y)\, dA.$$

$$= \int_1^2 \int_0^{2/x} y \cdot kx^2 \, dy \, dx$$

$$= \int_1^2 \left. kx^2 \cdot \frac{y^2}{2} \right]_0^{2/x} dx$$

$$= \int_1^2 2k \, dx$$

$$= 2k \, [x]_1^2 = 2k.$$

$$M_y = \iint_R x\, \rho(x,y)\, dA$$

$$= \int_1^2 \int_0^{2/x} x\, kx^2 \, dy \, dx = \int_1^2 kx^3 [y]_0^{2/x} dx$$

$$= \int_1^2 2k x^2 \, dx$$

$$= 2k \left(\frac{x^3}{3} \right)_1^2$$

$$= 2k(7/3) = \underline{\underline{\frac{14}{3} k}}$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{14}{3} k}{3k} = \frac{14}{9}$$

$$\bar{y} = \frac{M_x}{} = 2k = 2/$$

Centre of mass $= \left(\frac{14}{}, 2/ \right)$

Double integrals

Area $= \iint dA$ $\boxed{dA = dx\,dy}$

Volume $= \iint z\,dA$

Mass of Lamina $M = \iint \rho(x,y)\,dA$

$\rho \to$ density

Centre of mass $= (\bar{x}, \bar{y})$

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

$$M_y = \iint x\,\rho(x,y)\,dA$$

$$M_x = \iint y\,\rho(x,y)\,dA$$

Triple integrals

$$\text{Volume} = \iint dv$$

$$\boxed{dv = dx\,dy\,dz}$$

Polar co-ordinates.

Circle $x^2+y^2=r^2$

Put $x=r\cos\theta$, $y=r\sin\theta$, $dA=r\,dr\,d\theta$.

$r \to 0$ to r. (Eg: $x^2+y^2=4$
$r \to 0$ to 2)

$\theta \to 0$ to 2π.

Type 1 Region

$x \to$ constant, $y \to$ Variable

Type 2 - Region

$x \to$ Variable, y constant

4) Evaluate $\oint_C y^2 dx + x^2 dy$ where C is a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ oriented counter clockwise.

$\rightarrow$ Here $f = y^2$ $\quad g = x^2$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2(x-y)$$

By Green's thm $\int f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

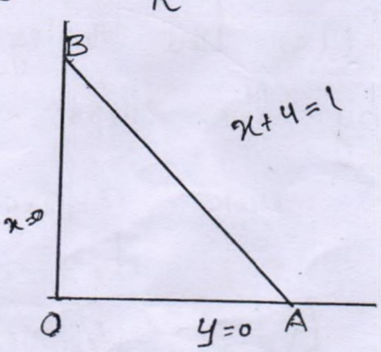
$$= \int_0^1 \int_0^1 2(x-y)\, dA$$

$$2 \int_0^1 \left[\frac{x^2}{2} - xy\right]_0^1 dy$$

$$= 2 \int_0^1 \frac{1}{2} - y \, dy = 2\left[\frac{y}{2} - \frac{y^2}{2}\right]_0^1 = 0$$

5) Verify Green's thm in the plane $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$

$\rightarrow$ Green's thm $\int f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$



L.H.S $\int_C = \int_{OA} + \int_{AB} + \int_{BO}$.

OA. $y=0$ $\quad dy=0$ $\quad x \to 0$ to 1

$$\int_{OA} f dx + g dy = \int_0^1 3x^2 dx = \left.\frac{3x^3}{3}\right]_0^1 = 1.$$

AB $\quad y = 1-x \quad dy = -dx \quad x \to 1$ to 0

$$\int_{AB} f dx + g dy = \int_1^0 [3x^2 - 8(1-x)^2] dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 3x^2 - 8(1-x)^2 - 4(1-x) + 6x - 6x^2\, dx$$

$$= \left[x^3 - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + \frac{6x^2}{2} - \frac{6x^3}{3}\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = 8/3$$

$\underline{\underline{BO}}$ $\quad x=0 \quad dx=0 \quad y \to 1 \text{ to } 0$

$$\int_{BO} f\,dx + g\,dy = \int_1^0 4y\,dy = \underline{\underline{-2}}$$

L.H.S $\quad \int_C f\,dx + g\,dy = 1 + \frac{8}{3} - 2 = \underline{\underline{\frac{5}{3}}}$.

R.H.S $\quad = \iint \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\, dA = \int_0^1 \int_0^{1-x} (-6y + 16y)\,dy\,dx$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx$$

$$= \int_0^1 10\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5\int_0^1 [1-x]^2\,dx = \left[5\,\frac{[1-x]^3}{-3}\right]_0^1 = \underline{\underline{\frac{5}{3}}}$$

L.H.S = R.H.S

Area Using Green's Theorem

$$\text{Area} = \frac{1}{2}\oint x\,dy - y\,dx.$$

Pbm

1) Use line integral to find the area enclosed by the ellipse.

$\to$ put $x = a\cos\theta \quad y = b\sin\theta$.

$dx = -a\sin\theta\,d\theta \quad dy = b\cos\theta\,d\theta \quad \theta \to 0 \text{ to } 2\pi$

$$\text{Area} = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} a\cos\theta \times b\cos\theta\,d\theta - b\sin\theta \times a\sin\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\left[\cos^2\theta + \sin^2\theta\right] d\theta.$$

$$\frac{ab}{2}\left[\theta\right]_0^{2\pi} = \underline{\pi ab}.$$

Applications of Green's theorem

1) We will use Green's thm to calculate the area bounded by the curve.

2) It is used to find the workdone of a force field.

3) Green's thm gives a relationship b/w the line integral of two dimensional vector field over a cld path in the plane and the double integral over the region it closes.

Surface Integrals

Let σ be a smooth parametric surface whose vector equation is $r = x(u,v)i + y(u,v)j + z(u,v)k$. Where (u,v) varies over a region R in the u-v plane. If $f(x,y,z)$ is continuous on σ, then

$$\iint_{\sigma} f(x,y,z)\, ds = \iint_{R} f(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

1) Let σ be the surface with equation $z = g(x,y)$ and let R be its projection on the xy plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ then

$$\iint_{\sigma} f(x,y,z)\, ds = \iint_{R} f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

2) Surface integral over $y = g(x,z)$, $R \rightarrow$ its projection on xz plane

$$\iint_{\sigma} f(x,y,z)\, ds = \iint_{R} f(x, g(x,z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

3) Surface integral over $x = g(y,z)$, R its projection of yz plane

$$\iint_{\sigma} f(x,y,z)\, ds = \iint_{R} f(g(y,z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA$$

Pbms

1) Evaluate the surface integral $\iint_\sigma xz\,ds$ where σ is the part of the plane $x+y+z=1$ that lies in the first octant.

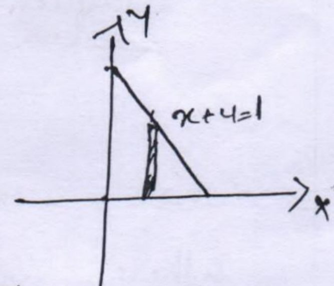
$\rightarrow$ $x+y+z=1 \implies z=1-x-y$ ie $Z=f(x,y)$.

$f(x,y,z)=xz$

$$\frac{\partial z}{\partial x}=-1 \qquad \frac{\partial z}{\partial y}=-1$$

$$\therefore \iint_\sigma xz\,ds = \iint_R x(1-x-y)\cdot\sqrt{(-1)^2+(-1)^2+1}\;dA$$

$$= \iint_R (x-x^2-xy)\sqrt{3}\;dA$$



R is the projection to xy plane

$$= \sqrt{3}\int_0^1\int_0^{1-x} [x-x^2-xy)\,dy\,dx.$$ $x \to 0$ to 1, $y \to 0$ to $1-x$.

$$= \sqrt{3}\int_0^1 \left. xy - x^2y - \frac{xy^2}{2}\right]_0^{1-x} dx$$

$$\left\{ \begin{aligned} & x(1-x)-x^2(1-x) - \frac{x(1-x)^2}{2}. \\ &= x - \cancel{x^2} - x^2+x^3 - \frac{x}{2} + \frac{2x^2}{2}. - \frac{x^3}{2} \\ &= \frac{x}{2} - x^2 + \frac{x^3}{2}. \end{aligned}\right.$$

$$= \sqrt{3}\int_0^1 \frac{x}{2} - x^2 + \frac{x^3}{2}\,dx$$

$$= \sqrt{3}\left[\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}$$

2) Evaluate the surface integral $\iint_\sigma y^2z^2\,ds$ Where σ is the part of the cone $Z=\sqrt{x^2+y^2}$ that lies b/w the planes $z=1$ & $z=2$.

$\rightarrow$ $Z=f(x,y)$ $\qquad ds = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\,2x = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$ds = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\;dA = \sqrt{2}\,dA$$

Pbms

1 Use the divergence thm to find the out-ward flux of the vector field $F(x,y,z) = z\bar{k}$ across the sphere $x^2+y^2+z^2=a^2$.

→ Let σ denote the outward-oriented spherical surface and G the region that it encloses. Then $\underline{\underline{\operatorname{div} F = 1}}$

$$\boxed{\text{Flux } \phi = \iint F\cdot n\, ds}$$

By divergence thm $\boxed{\iint F\cdot n\, ds = \iiint \operatorname{div} F\, dv}$

$$\therefore \phi = \iiint dv = \text{Volume of } G = \underline{\underline{\frac{4\pi a^3}{3}}}$$

2. Use the divergence thm to find the outward flux of the vector-field $F(x,y,z) = 2x\,i + 3y\,j + z^2 k$ across the unit cube.

⟹ $\operatorname{div} F = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = \underline{\underline{5+2z}}$.

$$\text{Flux } \phi = \iint F\cdot n\, ds = \iiint \operatorname{div} F\, dv = \iiint (5+2z)\, dv$$

$$= \int_0^1\int_0^1\int_0^1 [5+2z]\, dz\, dy\, dx$$

$$= \int_0^1\int_0^1 \left[5z+z^2\right]_0^1 dy\, dx.$$

$$= \int_0^1\int_0^1 6\, dy\, dx = \int_0^1 6(y)_0^1\, dx.$$

$$= 6\int_0^1 dx$$

$$= 6\,(x)_0^1 = \underline{\underline{6}}$$

Sources And Sinks

A point P in an incompressible fluid is said to be a source if $(\nabla \cdot F) > 0$ at P and it is said to be a sink if $(\nabla \cdot F)_P < 0$

If $(\nabla \cdot F)_P = 0$ then P is free of source and sink.

Proms

1 Determine whether the vector field $F = 4(x^3 - x)i + 4(y^3 - y)j + 4(z^3 - z)k$ is free of source and sink. If it is not locate them.

→ $\nabla \cdot F = 4(3x^2 - 1) + 4(3y^2 - 1) + 4(3z^2 - 1) = 12(x^2 + y^2 + z^2 - 1)$

free of source & sink $\nabla \cdot F = 0 \Rightarrow x^2 + y^2 + z^2 = 1$.

It is free of source and sink on the surface of sphere.

Source $\nabla \cdot F > 0$. ie $x^2 + y^2 + z^2 > 1$.

Sink $\nabla \cdot F < 0$ $x^2 + y^2 + z^2 < 1$.

2 Determine whether the vector field $F(x, y, z)$ is free of source and sinks. If it is not locate them.

(i). $F(x, y, z) = (y + z)i - xz^3 j + x^2 \sin y\, k$.

(ii) $F(x, y, z) = x^3 i + y^3 j + z^3 k$.

→

(i) $\nabla \cdot F = 0$ Hence no source or sinks

(ii) $\nabla \cdot F = 3x^2 + 3y^2 + 3z^2$

$\nabla \cdot F > 0$ for all pts except at origin.

Source at all pts except at the origin.

[$\nabla \cdot F$ cannot be negative. It is has no sinks]

3 Use Stoke's thm to evaluate $\int F \cdot dr$. Where $F(x,y,z) = xy\,i + yz\,j + zx\,k$: C is the triangle in the plane $x+y+z=1$ with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with a counter clockwise orientation looking from the first octant towards the origin

→ Stoke's thm $\int F \cdot dr = \iint_\sigma (\text{curl} F \cdot n)\, ds$.

$$\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \underline{-y\,i - z\,j - x\,k}$$

$x+y+z=1 \implies z = 1-x-y$ Hence.

$$n = -\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k = i+j+k$$

$$\text{curl} F \cdot n = [-y\,i - zj - xk] \cdot [i+j+k] = -y-z-x$$

$$\iint_\sigma \text{curl} F \cdot n\, ds = \iint_R (-y-z-x)\, dA$$

$$= \iint_R -y-(1-x-y)-x\, dA$$

$$= \iint_R -dA = -\iint_R dA$$

$$= -\frac{1}{2} \times 1 \times 1 = -1/2 \quad [\text{Area of a triangle}]$$

4. Consider the vector field given by the formula $F(x,y,z) = (x-z)\,i + (y-x)\,j + (z-xy)\,k$.

(1). Use Stoke's thm find the circulation around the triangle with vertices $A(1,0,0)$, $B(0,2,0)$ $C(0,0,3)$ oriented counter clockwise looking from the origin toward the first octant.

(ii). Find the circulation density of F at the origin in the direction of k

(iii) Find the unit vector n such that the circulation density of F at the origin is maximum in the direction of n.

(i). Equation of a plane passing through $A(1,0,0)$ $B(0,2,0)$ $C(0,0,1)$ is

$$2x+y+2z=2.$$

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & y-x & z-xy \end{vmatrix} = -xi+(y-1)j-k$$

Circulation $\int F\cdot T\,ds = \iint \text{curl}F\cdot n\,ds.$

$$2x+y+2z=2 \implies z = 1-x-y/2.$$

$$n = \frac{\partial z}{\partial x}i + \frac{\partial z}{\partial y}j - k.$$

$$= -i - \frac{j}{2} - k$$

$$\text{curl}F\cdot n = x - \frac{(y-1)}{2} + 1 = \underline{\underline{x - y/2 + 3/2}}$$

$$\iint_\sigma \text{curl}F\cdot n\,ds = \iint_R x - y/2 + 3/2\,dA$$

$$= \int_0^1 \int_0^{2-2x} x - \frac{y}{2} + \frac{3}{2}\,dy\,dx$$

$$= \int_0^1 x+2-3x^2\,dx = \underline{\underline{\frac{3}{2}}}.$$

(ii). Curl f at $(0,0,0) = -j - k$ also $n = k$

$$\text{curl} f \cdot n = \underline{-1}$$

(iii) The rotation of f has its maximum value at the origin about the unit vector in the direction of curl f $(0,0,0)$

so $n = \dfrac{-j-k}{\sqrt{1+1}} = \underline{-\dfrac{j}{\sqrt{2}} - \dfrac{k}{\sqrt{2}}}$

5 Use Stoke's thm to evaluate $\int_c f \cdot dr$ where

where $f = (x-y)i + (y-z)j + (z-x)k$ where C is the boundary of the portion of the plane $x+y+z=1$ in the first octant

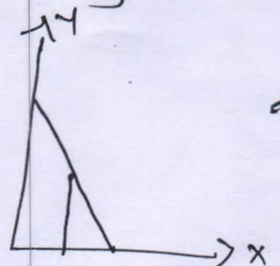
$$\int f \cdot dr = \iint \text{curl} f \cdot n \, ds.$$

$$\text{curl} f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \underline{i + j + k}$$

$z = 1 - x - y$

$$n = -\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k = \underline{i + j + k}$$

$$\text{curl} f \cdot n = [i + j + k] \cdot [i + j + k] = 1 + 1 + 1 = \underline{3}$$



$x \to 0$ to 1

$y \to 0$ to $\underline{1-x}$.

$$\iint \text{curl} f \cdot n \, ds = \int_0^1 \int_0^{1-x} 3 \, dA = 3 \int_0^1 (1-x) \, dx$$

$$= 3 \left[x - \frac{x^2}{2} \right]_0^1 = 3[1 - 1/2]$$

$$\underline{= 3/2}$$

Module II

Vector Integral theorems

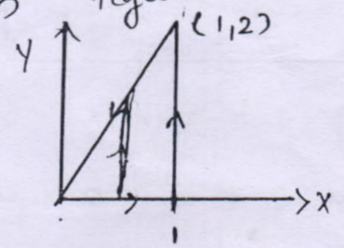
Green's Theorem

Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve 'C' oriented counter clockwise. If $f(x,y)$ and $g(x,y)$ are continuous and have continuous first partial derivatives on some open set containing R then,

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

Pbms

1) Use Green's thm evaluate $\oint x^2y\,dx + x\,dy$ along a triangular path shown in figure.



→ Here $f = x^2y$ $\quad g = x$.

$\frac{\partial f}{\partial y} = x^2 \quad \frac{\partial g}{\partial x} = 1$

$\therefore \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 1 - x^2$.

(0,0) to (1,2)

$\frac{y-0}{2-0} = \frac{x-0}{1-0}$

$y = 2x$

$\underline{y = 2x}$.

By Green's thm $\oint_C x^2y\,dx + x\,dy =$

$$\int_0^1 \int_0^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^1 \left[y - x^2y\right]_0^{2x} dx = \int_0^1 2x - 2x^3\,dx$$

$$= \left[\frac{2x^2}{2} - \frac{2x^4}{4}\right]_0^1$$

$$= \underline{1/2}.$$

2) Find the workdone by the force field

$$F(x,y) = [e^x - y^3]\,i + [\cos y + x^3]\,j$$

on a particle that travels once around the unit circle $x^2+y^2=1$ in the counterclockwise direction.

4) Evaluate $\oint_C y^2 dx + x^2 dy$ where C is a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ oriented counter clockwise.

$\rightarrow$ Here $f = y^2$ $\quad g = x^2$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial g}{\partial x} = 2x.$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2(x-y)$$

By Green's thm $\int f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$

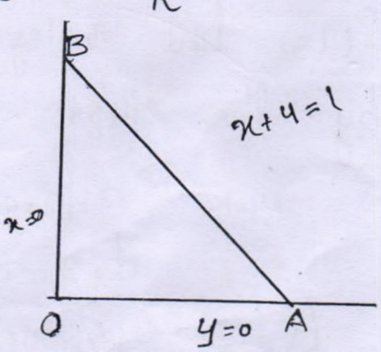
$$= \int_0^1 \int_0^1 2(x-y)\, dA$$

$$2 \int_0^1 \left[\frac{x^2}{2} - xy\right]_0^1 dy$$

$$= 2 \int_0^1 \frac{1}{2} - y\, dy = 2\left[\frac{y}{2} - \frac{y^2}{2}\right]_0^1 = 0$$

5) Verify Green's thm in the plane $\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where C is the boundary of the region defined by $x=0$, $y=0$, $x+y=1$

$\rightarrow$ Green's thm $\int f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$



L.H.S $\int_C = \int_{OA} + \int_{AB} + \int_{BO}$.

OA. $y=0$ $\quad dy=0$ $\quad x \to 0$ to 1

$$\int_{OA} f dx + g dy = \int_0^1 3x^2\, dx. = \left.\frac{3x^3}{3}\right]_0^1 = 1.$$

AB $\quad y = 1-x$ $\quad dy = -dx$ $\quad x \to 1$ to 0

$$\int_{AB} f dx + g dy = \int_1^0 [3x^2 - 8(1-x)^2]\, dx + [4(1-x) - 6x(1-x)]\, \bar{x}\, dx$$

$$= \int_1^0 3x^2 - 8(1-x)^2 - 4(1-x) + 6x - 6x^2 \, dx$$

$$= \left[x^3 - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + \frac{6x^2}{2} - \frac{6x^3}{3} \right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = 8/3$$

<u>BO</u> $\quad x=0 \quad dx=0 \quad y \to 1 \text{ to } 0$

$$\int_{BO} f\,dx + g\,dy = \int_1^0 4y\,dy = \underline{-2}$$

L.H.S $\quad \int_C f\,dx + g\,dy = 1 + \frac{8}{3} - 2 = \underline{\frac{5}{3}}$.

R.H.S $\quad : \iint \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \, dA = \int_0^1 \int_0^{1-x} (-6y + 16y) \, dy \, dx$

$$= \int_0^1 \int_0^{1-x} 10y \, dy \, dx$$

$$= \int_0^1 10 \left[\frac{y^2}{2} \right]_0^{1-x} dx$$

$$= 5 \int_0^1 [1-x]^2 \, dx = \left[5 \frac{[1-x]^3}{-3} \right]_0^1 = \underline{\frac{5}{3}}$$

L.H.S = R.H.S

<u>Area</u> <u>Using</u> <u>Green's</u> <u>Theorem</u>

$$\text{Area} = \frac{1}{2} \oint x\,dy - y\,dx.$$

<u>Pbm</u>

1) Use line integral to find the area enclosed by the ellipse.

$\rightarrow$ put $\quad x = a\cos\theta \quad y = b\sin\theta$.

$dx = -a\sin\theta\,d\theta \quad dy = b\cos\theta\,d\theta \quad \theta \to 0 \text{ to } 2\pi$

$$\text{Area} = \frac{1}{2} \oint x\,dy - y\,dx = \frac{1}{2} \int_0^{2\pi} a\cos\theta \times b\cos\theta\,d\theta - b\sin\theta \times -a\sin\theta\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} ab\left[\cos^2\theta + \sin^2\theta\right] d\theta.$$

$$\frac{ab}{2} [\theta]_0^{2\pi} = \underline{\pi ab}.$$

Applications of Green's theorem

1) We will use Green's thm to calculate the area bounded by the curve.
2) It is used to find the workdone of a force field.
3) Green's thm gives a relationship b/w the line integral of two dimensional vector field over a cld path in the plane and the double integral over the region it closes.

Surface Integrals

Let σ be a smooth parametric surface whose vector equation is $r = x(u,v)i + y(u,v)j + z(u,v)k$. Where (u,v) varies over a region R in the u-v plane. If $f(x,y,z)$ is continuous on σ, then

$$\iint_{\sigma} f(x,y,z)\,ds = \iint_{R} f(x(u,v), y(u,v), z(u,v)) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

1) Let σ be the surface with equation $z = g(x,y)$ and let R be its projection on the xy plane. If g has continuous first partial derivatives on R and $f(x,y,z)$ is continuous on σ then

$$\iint_{\sigma} f(x,y,z)\,ds = \iint_{R} f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

2) Surface integral over $y = g(x,z)$, $R \rightarrow$ its projection on xz plane

$$\iint_{\sigma} f(x,y,z)\,ds = \iint_{R} f(x, g(x,z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

3) Surface integral over $x = g(y,z)$, R its projection of yz plane

$$\iint_{\sigma} f(x,y,z)\,ds = \iint_{R} f(g(y,z), y, z) \sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA$$

Pbms

1) Evaluate the surface integral $\iint_\sigma xz\, ds$ where σ is the part of the plane $x+y+z=1$ that lies in the first octant.

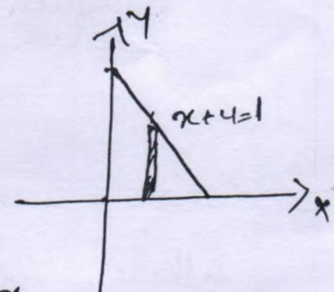
$\longrightarrow$ $x+y+z=1 \implies z=1-x-y$ ie $z=f(x,y)$.

$f(x,y,z)=xz$

$$\frac{\partial z}{\partial x}=-1 \qquad \frac{\partial z}{\partial y}=-1$$

$$\therefore \iint_\sigma xz\, ds = \iint_R x(1-x-y)\cdot\sqrt{(-1)^2+(-1)^2+1}\; dA$$

$$= \iint_R (x-x^2-xy)\sqrt{3}\; dA$$



R is the projection to xy plane

$$= \sqrt{3}\int_0^1\int_0^{1-x} (x-x^2-xy)\, dy\, dx.$$

$x \to 0$ to 1

$y \to 0$ to $1-x$.

$$= \sqrt{3}\int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_0^{1-x} dx$$

$$\left\{ \begin{aligned} & x(1-x)-x^2(1-x) - \frac{x(1-x)^2}{2}. \\ &= x-\cancel{x^2}-x^2+x^3 - \frac{x}{2}+\frac{2\cancel{x^2}}{2}-\frac{x^3}{2} \\ &= \frac{x}{2}-x^2+\frac{x^3}{2}. \end{aligned}\right.$$

$$= \sqrt{3}\int_0^1 \frac{x}{2}-x^2+\frac{x^3}{2}\, dx$$

$$= \sqrt{3}\left[\frac{x^2}{4}-\frac{x^3}{3}+\frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}$$

---

2) Evaluate the surface integral $\iint_\sigma y^2z^2\, ds$ where σ is the part of the cone $z=\sqrt{x^2+y^2}$ that lies b/w the planes $z=1$ & $z=2$.

$\longrightarrow$ $z=f(x,y)$ $\qquad ds = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\; dA$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}\, 2x = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$ds = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\; dA = \sqrt{2}\, dA$$

Pbms

1 Use the divergence thm to find the out-ward flux of the vector field $F(x,y,z) = z\bar{k}$ across the sphere $x^2+y^2+z^2=a^2$.

→ Let σ denote the outward-oriented spherical surface and G the region that it encloses. then $\underline{\underline{\operatorname{div} F = 1}}$

$$\boxed{\text{Flux } \phi = \iint F \cdot n \, ds}$$

By divergence thm $\boxed{\iint F \cdot n \, ds = \iiint \operatorname{div} F \, dv}$

$$\therefore \phi = \iiint dv = \text{Volume of } G = \underline{\underline{\frac{4\pi a^3}{3}}}$$

2. Use the divergence thm to find the outward flux of the vector-field $F(x,y,z) = 2x\,i + 3y\,j + z^2 k$. across the unit cube,

⟹ $\operatorname{div} F = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(z^2) = \underline{\underline{5+2z}}$.

$$\text{Flux } \phi = \iint F \cdot n \, ds = \iiint \operatorname{div} F \, dv = \iiint (5+2z)\, dv$$

$$= \int_0^1 \int_0^1 \int_0^1 [5+2z] \, dz\,dy\,dx$$

$$= \int_0^1 \int_0^1 \left[5z + z^2\right]_0^1 dy\,dx.$$

$$= \int_0^1 \int_0^1 6 \, dy\,dx = \int_0^1 6(y)_0^1 \, dx.$$

$$= 6 \int_0^1 dx$$

$$= 6\,(x)_0^1 = \underline{\underline{6}}$$

Sources and Sinks

A point P in an incompressible fluid is said to be a source if $(\nabla \cdot F)_P > 0$ at P and it is said to be a sink if $(\nabla \cdot F)_P < 0$

If $(\nabla \cdot F)_P = 0$ then P is free of source and sink.

Probs

1 Determine whether the vector field $F = 4(x^3 - x)i + 4(y^3 - y)j + 4(z^3 - z)k$ is free of source and sink. If it is not locate them.

→ $\nabla \cdot F = 4(3x^2 - 1) + 4(3y^2 - 1) + 4(3z^2 - 1) = 12(x^2 + y^2 + z^2 - 1)$

Free of source & sink $\nabla \cdot F = 0 \Rightarrow x^2 + y^2 + z^2 = 1$.

It is free of source and sink on the surface of sphere.

Source $\nabla \cdot F > 0$. ie $x^2 + y^2 + z^2 > 1$.

Sink $\nabla \cdot F < 0$ $x^2 + y^2 + z^2 < 1$.

2 Determine whether the vector field $F(x, y, z)$ is free of source and sinks. If it is not locate them.

(i). $F(x, y, z) = (y + z)i - xz^3 j + x^2 \sin y\, k$.

(ii) $F(x, y, z) = x^3 i + y^3 j + z^3 k$.

→

(i) $\nabla \cdot F = 0$ Hence no source or sinks

(ii) $\nabla \cdot F = 3x^2 + 3y^2 + 3z^2$

$\nabla \cdot F > 0$ for all pts except at origin.

Source at all pts except at the origin.

[$\nabla \cdot F$ cannot be negative. It has no sinks]

3 Use Stoke's thm to evaluate $\int F \cdot dr$. Where $F(x,y,z) = xy\,i + yz\,j + zx\,k$: C is the triangle in the plane $x+y+z=1$ with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with a counter clockwise orientation looking from the first octant towards the origin

→ Stoke's thm $\int F \cdot dr = \iint_\sigma (\text{curl}F \cdot n)\, ds$.

$$\text{curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \underline{\underline{-y\,i - z\,j - x\,k}}$$

$x+y+z=1 \implies z = 1-x-y$ Hence.

$$n = -\frac{\partial z}{\partial x}\, i - \frac{\partial z}{\partial y}\, j + k = i+j+k$$

$$\text{curl}F \cdot n = [-y\,i - z\,j - x\,k] \cdot [i+j+k] = -y-z-x$$

$$\iint_\sigma \text{curl}F \cdot n\, ds = \iint_R (-y-z-x)\, dA$$

$$= \iint_R -y-(1-x-y)-x\, dA$$

$$= \iint_R -dA = -\iint_R dA$$

$$= -\frac{1}{2} \times 1 \times 1 = -\frac{1}{2} \quad [\text{Area of a triangle}]$$

4. Consider the vector field given by the formula $F(x,y,z) = (x-z)\,i + (y-x)\,j + (z-xy)\,k$.

(i). Use Stoke's thm find the circulation around the triangle with vertices $A(1,0,0)$, $B(0,2,0)$ $C(0,0,3)$ oriented counter clockwise looking from the Origin toward the first Octant.

(ii). Find the circulation density of F at the Origin in the direction of k

(iii) Find the unit vector n such that the circulation density of F at the Origin is maximum in the direction of n.

(i). Equation of a plane passing through $A(1,0,0)$ $B(0,2,0)$ $C(0,0,1)$ is

$2x+y+2z=2.$

$$\text{Curl}F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-z & y-x & z-xy \end{vmatrix} = -xi+(y-1)j-k$$

Circulation $\int F\cdot T\,ds = \iint \text{curl}F\cdot n\,ds.$

$2x+y+2z=2 \implies z = 1-x-y/2.$

$$n = \frac{\partial z}{\partial x}i + \frac{\partial z}{\partial y}j - k.$$

$$= -i - \frac{j}{2} - k$$

$$\text{Curl}F\cdot n = x - \frac{(y-1)}{2} + 1 = \underline{x - y/2 + 3/2}$$

$$\iint_\sigma \text{curl}F\cdot n\,ds = \iint_R x-y/2+3/2\,dA$$

$$= \int_0^1 \int_0^{2-2x} x-\frac{y}{2}+\frac{3}{2}\,dy\,dx$$

$$= \int_0^1 x+2-3x^2\,dx = \underline{\frac{3}{2}}.$$

(ii). curl f at (0,0,0) = $-j-k$ also $n = k$

$\therefore$ curl f · n = $\underline{\underline{-1}}$

(iii) The rotation of f has its maximum value at the origin about the unit vector in the direction of curl f (0,0,0)

So $n = \dfrac{-j-k}{\sqrt{1+1}} = \underline{\underline{-\dfrac{j}{\sqrt{2}} - \dfrac{k}{\sqrt{2}}}}$

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5 Use Stoke's thm to evaluate $\int_C f \cdot dr$ where $f = (x-y)i + (y-z)j + (z-x)k$ where C is the boundary of the portion of the plane $x+y+z=1$ in the first octant

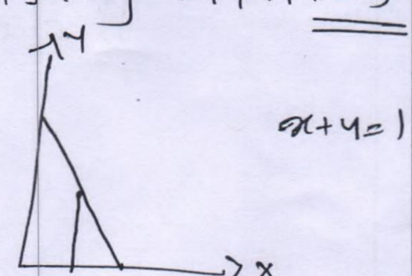
→ $\int f \cdot dr = \iint \text{curl} f \cdot n \, ds.$

$$\text{curl} f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & y-z & z-x \end{vmatrix} = \underline{\underline{i+j+k}}$$

$Z = 1-x-y$

$$n = -\frac{\partial z}{\partial x} i - \frac{\partial z}{\partial y} j + k = \underline{\underline{i+j+k}}$$

$$\text{curl} f \cdot n = [i+j+k] \cdot [i+j+k] = \underline{\underline{1+1+1=3}}$$



$x \to 0$ to 1

$y \to 0$ to $\underline{\underline{1-x}}$

$$\iint \text{curl} f \cdot n \, ds = \int_0^1 \int_0^{1-x} 3 \, dA = 3\int_0^1 (1-x)\, dx$$

$$= 3\left[x - \frac{x^2}{2}\right]_0^1 = 3[1-1/2]$$
$$\underline{\underline{= 3/2}}$$